Math 620, Fall, 1999
Homework Set 2: Dedekind Domains
due Friday, September 24, 1999

1. (basically Janucsz, Exercise 4, p. 19) Let R be a Dedekind domain and let S be a multiplicative set in R. Show that the localization R_S is also a Dedekind domain, and that the natural map of class groups $C(R) \to C(R_S)$ is surjective. (Hint: What are the fractional ideals of R_S?)

2. Let $R = \mathbb{Z}[\sqrt{-3}]$, with field of fractions $F = \mathbb{Q}[\sqrt{-3}]$. From the last homework set, R is not integrally closed in F, and hence is not a Dedekind domain. Exhibit a fractional ideal in R that does not have an inverse. Is this fractional ideal a projective R-module?

3. (close to Janucsz, Exercise 5, p. 19) Let $R = \mathbb{Z}[\sqrt{-5}]$, which by the last homework set is the integral closure of $\mathbb{Z}$ in $F = \mathbb{Q}[\sqrt{-5}]$. We will see that R is a Dedekind domain.

(a) Let $\mathfrak{p} = (3, 2 + \sqrt{-5})$. Show that $R/\mathfrak{p}$ is a field of three elements and thus that $\mathfrak{p}$ is a maximal ideal of R.

(b) Show that $\mathfrak{p}$ is not principal. Hint: show that neither of the generators divides the other, and that if there were a single generator $a + b\sqrt{-5}$ with a and b in $\mathbb{Z}$, then one would have to have $a^2 + 5b^2 = 3$, which is impossible.

(c) Show that $\mathfrak{p}$ is an element of order 2 in $C(R)$ by showing that $\mathfrak{p}^2 = (2 + \sqrt{-5})$.

4. Let $R = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$, the ring of real-valued polynomial functions on the circle $x^2 + y^2 = 1$ in the x-y plane. Show that R is a Dedekind domain. (Hint: Obviously R is Noetherian. Show that every non-zero prime ideal is maximal and that R is integrally closed in its field of fractions.) For extra credit, but hard: See if you can show $C(R)$ has order 2, by finding all the maximal ideals and determining which ones are principal.