Math 620, Fall, 1999
Homework Set 5: Consequences of Minkowski's Theorem and Related Topics
due Friday, November 5, 1999

1. Let $R = \mathbb{Z}[\sqrt{-5}]$, the ring of integers in the field $\mathbb{Q}(\sqrt{-5})$.
   (a) Show that $\mathfrak{q} = (2, 1+\sqrt{-5})$ is a prime ideal in R with $\mathfrak{q}^2 = (2)$. (Hint: $N_{K/\mathbb{Q}}(\mathfrak{q}) \supseteq (N_{K/\mathbb{Q}}(2), N_{K/\mathbb{Q}}(1+\sqrt{-5})) = (4,6) = (2)$.)
   (b) Show that $\mathfrak{p} = (3, 2+\sqrt{-5})$ is a prime ideal in R with $\mathfrak{p}\bar{\mathfrak{p}} = (3)$. Here $\bar{\mathfrak{p}} = (3, 2-\sqrt{-5})$ is the complex conjugate ideal.
   (c) Show that $\mathfrak{q}$, $\mathfrak{p}$, and $\bar{\mathfrak{p}}$ all lie in the same class in $C(R)$, and thereby complete the proof begun in class that $C(R)$ is of order 2.

2. Do Exercise 1 in Janusz, p. 73, which shows that over a number field other than $\mathbb{Q}$, it is possible to have an extension in which no primes ramify.

3. This problem relates to the field $K = \mathbb{Q}(\sqrt{5})$.
   (a) Show that K has class number 1, i.e., that the ring of integers R in K is a PID. (Imitate the argument in problem 1.)
   (b) What prime numbers $p \in \mathbb{N}$ can be written in the form $x^2 - 5y^2$, x, $y \in \mathbb{Z}$? (Hint: $x^2 - 5y^2 = N_{K/\mathbb{Q}}(x + y\sqrt{5})$. First see for what primes $\pm p$ can be written in the indicated form; then try to resolve the question of signs.)