Math 620, Fall, 1999
Homework Set 6: $\mathfrak{p}$-adic Fields (Janusz, §II.2)
Selected Solutions

3. Show that the power series for $(1-4z)^{\frac{1}{2}}$ has integer coefficients ... .
Solution (Suggested by Mr. Ren). Clearly the series has the form

$$(1-4z)^{\frac{1}{2}} = 1 - 2z + \sum_{i=2}^{\infty} a_i z^i,$$

where

$$\left(1 - 2z + \sum_{i=2}^{\infty} a_i z^i\right)^2 = 1 - 4z.$$

When we expand out the left-hand side, the coefficient of z^i must vanish for $i \geq 2$. So we obtain the recurrence relations

$$\begin{cases} 2a_i - 4a_{i-1} + 2a_{i-2}a_2 + \cdots + 2a_{(i+2)/2}a_{(i-2)/2} + a_{i/2}^2 = 0, & i \text{ even}, \\ 2a_i - 4a_{i-1} + 2a_{i-2}a_2 + \cdots + 2a_{(i+1)/2}a_{(i-1)/2} = 0, & i \text{ odd}. \end{cases}$$

From these equations it is easy to prove by induction on i that each a_i is an even integer. (When we solve for a_i, we have to divide by 2, but one checks that the quantity to be divided by 2 is in fact $\equiv 0 \bmod 4$.) For example, one has $a_2 = -4/2 = -2$, $a_3 = 4a_2/2 = -4$, $a_4 = (4a_3 - a_2^2)/2 = (-16-4)/2 = -10$, etc.

7. Show that the $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ has order 8 if $p = 2$.
Solution. We have $\mathbb{Q}_2^\times \cong \{2^n : n \in \mathbb{Z}\} \times \mathbb{Z}_2^\times$, where $\mathbb{Z}_2^\times$ is the group of units in $\mathbb{Z}_2$, and so $(\mathbb{Q}_2^\times) \cong \{2^{2n} : n \in \mathbb{Z}\} \times (\mathbb{Z}_2^\times)^2$. Now each element of $\mathbb{Z}_p^\times$ has an expansion $1 + a_1 \cdot 2 + a_2 \cdot 4 + a_3 \cdot 8 + \cdots$, where $a_j \in \{0,1\}$. Now

$$(1 + a_1 \cdot 2 + a_2 \cdot 4 + a_3 \cdot 8 + \cdots)^2 = 1 + b_3 \cdot 8 + \cdots,$$

so $(\mathbb{Z}_p^\times)^2 \subseteq V$, where $V = \{x \in \mathbb{Z}_2 : x \equiv 1 \pmod 8\}$. But in the other direction, $V \subseteq (\mathbb{Z}_2^\times)^2$ by an application of problem 3. So

$$\mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2 \cong \mathbb{Z}/2 \times (\mathbb{Z}_2^\times/V) \cong \mathbb{Z}/2 \times (\mathbb{Z}/8)^\times.$$

But $(\mathbb{Z}/8)^\times$ is non-cylic of order 4 (with elements represented by 1, 3, 5, and 7). So $\mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$, which has order 8.