MATH 632, Homework #3: An application of Hilbert Spaces and the Riesz Representation Theorem: The Bergman Kernel

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Let $\Omega \subset \mathbb{C}$ be a non-empty connected open set in the complex plane, and define the *Bergman space* of Ω, $A^2(\Omega)$, to be the subspace of $L^2(\Omega)$ (with respect to Lebesgue measure in the plane) consisting of L^2 analytic functions. Of course, this space may be empty (there are no L^2 analytic functions on all of $\mathbb{C}$), but if Ω is bounded, it clearly contains the analytic functions that extend to be continuous on the closure of Ω.

1. Show that $A^2(\Omega)$ is a Hilbert space. Since, by definition, it is a subspace of $L^2(\Omega)$ and you may assume L^2 is complete, the only issue is to show that an L^2-limit of analytic functions is analytic.

2. Show that if $z \in \Omega$, then $f \mapsto f(z)$ is a bounded linear functional on $A^2(\Omega)$. (Hint: by the mean value property of analytic functions, if f is analytic, then $f(z)$ is the average of f over a small disk D_z centered at z, so
$$f(z) = \frac{1}{\text{area}(D_z)} \iint_{D_z} f(z)\, dx\, dy,$$
where $z = x + iy$ and $dx\, dy$ is Lebesgue measure. This can be rewritten to say that $f(z) = \langle f, g_z \rangle$ for a suitable L^2-function g_z.)

3. Deduce that there is a function $K(z, z')$, in $A^2(\Omega)$ as a function of z for fixed z', such that
$$f(z') = \langle f, K(\cdot, z') \rangle = \iint_{\Omega} f(z)\overline{K(z, z')}\, dx\, dy.$$

4. Now specialize to the case when Ω is the unit disk in $\mathbb{C}$. Compute an explicit orthonormal basis for $A^2(\Omega)$. (Apply the Gram-Schmidt process to 1, z, z^2, $\cdots$. It helps to normalize the Lebesgue measure by using $\frac{1}{\pi}\, dx\, dy$ in place of $dx\, dy$; that way, Ω has total area 1.) Finally, compute (explicitly) the Bergman kernel K.