MATH 632, Homework #4: More on Duality of Banach Spaces and Weak-$*$ Convergence

Prof. Jonathan Rosenberg

due Monday, October 16, 2006

Let $C(S^1)$ be the space of continuous functions on the circle, and let $C^1(S^1)$ be the space of C^1 functions on the circle. The latter consists of functions whose derivative is continuous. In this problem, we identify functions on S^1 with functions on $[0, 1]$ with the same values at 0 at as 1.

1. Show that $C^1(S^1)$ is a Banach space under the norm
$$\|f\| = \|f\|_\infty + \|f'\|_\infty,$$
where $\| \ \|_\infty$ denotes the usual sup norm for continuous functions.

2. There is an obvious continuous map $f \mapsto f'$ from $C^1(S^1)$ to $C(S^1)$. Show that the kernel of this map is one-dimensional and that the image is of codimension 1. Use this to show that
$$\|f\|' = |f(0)| + \|f'\|_\infty$$
is an equivalent norm on $C^1(S^1)$, and that a linear functional λ on $C^1(S^1)$ is continuous if and only if its restriction to functions vanishing at 0 is given by $f \mapsto \int f' \, d\mu$, for some measure μ on S^1. Show also that μ is determined only modulo multiples of Lebesgue measure.

3. Following the same idea as in Lax's proof (Chapter 11) that there is a continuous function whose Fourier series does not converge everywhere, study the pointwise convergence of Fourier series for functions in $C^1(S^1)$. By #2 above, it suffices to restrict attention to functions which are 0 at 0, and to study the linear functionals λ_n given by taking the n-th partial sum of the Fourier series, evaluated at 0. Use integration by parts to rewrite λ_n in the standard form $f \mapsto \int f' \, d\mu_n$, and show that $\|\mu_n\| \to 0$, and deduce that the Fourier series of a function in $C^1(S^1)$ *does* converge pointwise at 0. (Note that you are showing that if $f \in C^1(S^1)$ and $f(0) = 0$, then $\lambda_n(f) \to 0$. Then you get the general case by adding in a constant.) Then since there is nothing special about the point 0 (just change coordinates), deduce that the Fourier series of a C^1 function converges pointwise to the function.

4. Show, however, that the Fourier series of a C^1 function does *not* necessarily converge to the function in the C^1 topology.