MATH 632, Homework #5: Compact Convex Sets and the Krein-Milman Theorem

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1. Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space, and let $K = \{x \in \mathcal{H} : \|x\| \leq 1\}$. Recall that the set $\mathcal{E}$ of extreme points in K is the unit sphere, $\{x \in \mathcal{H} : \|x\| = 1\}$. Show that $\mathcal{E}$ is dense in K in the weak topology. Hint: It suffices to show that if $e_1, e_2, \cdots$ is an orthonormal basis, then ce_1 is in the weak closure of $\mathcal{E}$ for $0 \leq c \leq 1$. Why? Show that $ce_1 + \sqrt{1-c^2}e_j \in \mathcal{E}$ for $j \geq 2$, and converges to ce_1 weakly as $j \to \infty$.

2. Let G be a discrete group, and let $\mathcal{P}$ be the set of [normalized] "positive definite functions" on G. These are functions $\varphi\colon G \to \mathbb{C}$ with the properties that $\varphi(1_G) = 1$ (where 1_G denotes the identity element of G), $\varphi(x^{-1}) = \overline{\varphi(x)}$, and the matrix $\{\varphi(x_i^{-1}x_j)\}_{i,j\leq n}$ (which is hermitian symmetric with 1s on the diagonal) has all its eigenvalues ≥ 0, for any finite subset $x_1, \cdots, x_n$ of G. The interest in such functions comes from the observation that a function φ lies in $\mathcal{P}$ if and only if there is a unitary representation π of G on a Hilbert space $\mathcal{H}$, that is, a group homomorphism π from G to the unitary operators on $\mathcal{H}$, and a vector $\xi \in \mathcal{H}$ with $\|\xi\| = 1$, such that
$$\varphi(x) = \langle \pi(x)\xi, \xi \rangle. \qquad (1)$$
(You may take this assertion on faith, even though it is not so difficult to prove.)

   (a) Show that $\mathcal{P}$ consists of functions with sup norm 1. (Hint: Compute the eigenvalues of the matrix $\{\varphi(x_i^{-1}x_j)\}_{i,j\leq 2}$ when $x_1 = 1_G$, $x_2 = x$.)

   (b) Show that $\mathcal{P}$ is a compact convex set in $L^1(G)' = L^\infty(G)$ (equipped with the weak-$*$ topology). Its extreme points are called *characters*; they correspond to triples $(\mathcal{H}, \pi, \xi)$ as above with π *irreducible*, that is, not the direct sum of two non-zero smaller $\pi(G)$-invariant subspaces. Verify this by showing that if $(\mathcal{H}, \pi)$ is decomposable as a direct sum, and ξ splits as $\xi_1 \oplus \xi_2$ with ξ_j both non-zero, then the corresponding positive definite function defined in (1) is *not* extreme.

   (c) Now deduce that the Krein-Milman Theorem implies the *Gelfand-Raikov Theorem*, that G has enough irreducible unitary representations to separate points. (Hint: You need to show that given $x_1 \neq x_2$ in G, there is a character with

$\varphi(x_1^{-1}x_2) \neq 1$. By Krein-Milman, it's enough to find a function $\varphi \in \mathcal{P}$ with this property. But this is easy—show that the function that is 1 at 1_G and 0 elsewhere does the trick.)