MATH 632, Homework #6: Distributions, The Fourier Transform, and Linear Operators

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Let $\mathbb{T}$ denote the unit circle in $\mathbb{C}$, which is a compact abelian group with respect to multiplication, and let $\mathbb{T}^n$ denote the n-fold product if $\mathbb{T}$ with itself, the n-torus. For α a multi-index, let D^α denote the partial differentiation operator

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_1}}{\partial z_n^{\alpha_n}},$$

where if $\alpha_j < 0$, $\partial^{\alpha_j}/\partial z_j^{\alpha_j}$ is to be interpreted as $\partial^{|\alpha_j|}/\partial \overline{z_j}^{|\alpha_j|}$. Let $\mathcal{S}(\mathbb{T}^n)$ denote $C^\infty(\mathbb{T}^n)$ with its usual Fréchet topology, in which a sequence of functions $\{f_n\}$ converges to f if $D^\alpha f_n \to D^\alpha f$ uniformly for all multi-indices α.

1. Show that the ring $\mathbb{C}[z_1^\pm, \cdots, z_n^\pm]$ of trigonometric polynomials (finite linear combinations of functions $z \mapsto z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, with $\alpha \in \mathbb{Z}^n$) is dense in $\mathcal{S}(\mathbb{T}^n)$. (This is a strengthening of the Weierstrass theorem, since if $f \in \mathcal{S}(\mathbb{T}^n)$, you need to approximate not only f, but also all its derivatives, uniformly. Hint: given N, use the Weierstrass theorem to approximate all derivatives through order N to within $1/N$, then let N go to infinity.)

2. Let $\mathcal{S}(\mathbb{Z}^n)$ denote the set of rapidly decreasing sequences $\{c_\alpha\}_{\alpha\in\mathbb{Z}^n}$ indexed by $\mathbb{Z}^n$. The "rapidly decreasing" condition means that for any integer k, $(1 + |\alpha|^2)^{k/2} c_\alpha$ is bounded in α. Show that $\mathcal{S}(\mathbb{Z}^n)$ has a natural Fréchet space topology.

3. For $f \in \mathcal{S}(\mathbb{T}^n)$, define its *Fourier transform* to be the sequence $\{c_\alpha\}_{\alpha\in\mathbb{Z}^n}$ given by

$$(\mathcal{F}f)(\alpha) = c_\alpha = \int_{\mathbb{T}^n} f(z) z_1^{-\alpha_1} \cdots z_n^{-\alpha_n}\, dz,$$

   where dz is normalized Lebesgue measure with total mass 1. (Thus when $n = 1$, $dz = \frac{1}{2\pi} d\theta$ if $z = e^{i\theta}$.) Imitating the proof we gave in class in the case of $\mathbb{R}^n$, show that the Fourier transform $\mathcal{F}$ is a continuous linear map from $\mathcal{S}(\mathbb{T}^n)$ to $\mathcal{S}(\mathbb{Z}^n)$.

4. Prove the Fourier inversion formula: that $\mathcal{F}$ is a topological isomorphism and that for $f \in \mathcal{S}(\mathbb{T}^n)$,

$$f(z) = \sum_\alpha (\mathcal{F}f)(\alpha) z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

Hint: This is much easier than in the case of $\mathbb{R}^n$. Simply prove that the formula holds for $f \in \mathbb{C}[z_1^{\pm}, \cdots, z_n^{\pm}]$ and use part (1).

5. Deduce by duality that the Fourier transform gives a topological isomorphism from the space $\mathcal{S}'(\mathbb{T}^n)$ of distributions on $\mathbb{T}^n$ to a Fréchet space $\mathcal{S}'(\mathbb{Z}^n)$ of "tempered sequences." Identify the latter space explicitly.

6. A constant-coefficient partial differential operator T on $\mathbb{T}^n$ is simply a finite linear combination of the D^α's. Show that via Fourier transform, such an operator can be identified with multiplication by a polynomial in $\mathcal{F}T \in \mathbb{C}[z_1^{\pm}, \cdots, z_n^{\pm}]$ on $\mathcal{S}(\mathbb{T}^n)$. Deduce that T is injective if and only if $\mathcal{F}T$ does not vanish on the lattice $\mathbb{Z}^n$.

7. Show that it is possible for T as in (5) to be injective but not surjective on $\mathcal{S}(\mathbb{T}^n)$, but that this phenomenon doesn't happen if $n = 1$. (Hint: Diophantine approximation.)