MATH 632, Homework #7:
Banach Algebras and
Spectral Theory of Linear Operators

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due Wednesday, December 6, 2006

1. ("Generalized Primary Decomposition") Let X be a Banach space and let $T : X \to X$ be a bounded linear operator with disconnected spectrum $\sigma(T) = A \amalg B$. (This notation means the disjoint union of two disjoint closed subsets A and B in $\mathbb{C}$. The spectrum is to be computed in the Banach algebra $\mathcal{L}(X)$ of bounded linear operators on X, equipped with the operator norm.) Show that there is a decomposition $X = Y \oplus Z$ of X as a direct sum of two closed subspaces, so that T maps Y into Y, Z into Z, and $\sigma(T|_Y) = A$, $\sigma(T|_Z) = B$. Explain how the "primary decomposition theorem" in undergraduate linear algebra is the special case where X is finite dimensional. (Hint: The function f which is identically 1 on A and identically 0 on B is holomorphic on $\sigma(T)$, so you can apply the holomorphic functional calculus to obtain $P = f(T)$, which will be projection onto Y. Projection onto Z is obtained as $1 - P$.)

2. Let A be $L^1([0, \infty))$ (with respect to Lebesgue measure) and make A into a Banach algebra under convolution:
$$(f \star g)(x) = \int_0^x f(y)g(x-y)\,dy.$$
Check that A is a commutative nonunital Banach algebra. Show that the Fourier transform of each function $f \in A$ has a holomorphic extension to the upper half-plane, and use this to determine $\widehat{A}$. What is the closure of the image of A under the Gelfand transform?