MATH 632, Final Homework Set / Take-Home Final

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due Monday, December 18, 2006

Important Note: The assignment covers topics from various parts of the course as a way for you to "put things together." It counts twice as much as a normal homework assignment (20 points instead of 10) even though it's not any harder or much longer. It's very important that you get it in by the due date (the day when the final exam would have taken place had there been one) since I need to turn grades in by Wednesday the 20th and I need time to do the grading.

1. (concerning weak-$*$ convergence) Let μ be normalized Lebesgue measure $\frac{d\theta}{2\pi}$ on the circle $\mathbb{T}$, and let $\alpha \in [0, 1]$ be an irrational number. For any $z \in \mathbb{T}$, let δ_z be the point mass at z.

   (a) Show that as $n \to \infty$,
   $$\frac{1}{n} \sum_{j=0}^{n-1} \delta_{e^{2\pi i j \alpha}} \longrightarrow \mu$$
   in the weak-$*$ topology of measures (i.e., in the $\sigma(C(\mathbb{T})', C(\mathbb{T}))$ topology).

   (b) Show that one *cannot* get a similar result for convergence in norm, by showing that if ν is a finite convex combination $\nu = \sum_j t_j \delta_{z_j}$ of point measures, $z_j \in \mathbb{T}$, $\sum_j t_j = 1$, $t_j \geq 0$ for all j, then $\|\mu - \nu\| = 2$.

2. (concerning closed convex sets)

   (a) Show that any unit vector in the closed unit ball of a Hilbert space is an extreme point of the closed unit ball.

   (b) Show that the closed unit ball of $L^1([0, 1])$ (you may take this to mean the L^1 space of *real-valued* functions) contains no extreme points at all.

   (c) Deduce from (b) that $L^1([0, 1])$ is not isometrically isomorphic to the dual space of a Banach space.

3. (concerning Toeplitz and compact operators) Let $\mathcal{H}$ be the Hilbert space $L^2(\mathbb{T})$ (with respect to normalized Lebesgue measure $\frac{d\theta}{2\pi}$. Recall (you don't need to reprove this) that $\mathcal{H}$ has $\{z^n\}_{n\in\mathbb{Z}}$ as an orthonormal basis. Let P be orthogonal projection onto the closed linear span H^2 of $\{z^n\}_{n\geq 0}$, so that $1-P$ is orthogonal projection onto the closed linear span of $\{z^n\}_{n<0}$. Let $f \in C(\mathbb{T})$ and let M_f be the operator of multiplication by f on $\mathcal{H}$.

(a) Show that $\|M_f\| = \|f\|$ (the usual sup norm on $C(\mathbb{T})$).

(b) Show that $(1-P)M_f P$ is compact. (Hint: First suppose f has finite Fourier series, i.e., is a Laurent polynomial in z. Then do the general case by approximation.) Deduce that the commutator $[P, M_f] = PM_f - M_f P$ is compact.

(c) Let T_f be the *Toeplitz operator* PM_fP, viewed as a bounded operator on $H^2 = \operatorname{range} P$. Show that $(T_f)^* = T_{\bar{f}}$ and that $f \mapsto T_f$ is multiplicative modulo compacts, i.e., that $T_{fg} - T_f T_g \in \mathcal{K}(H^2)$. (Use (b).)

(d) Show that $T_{z^{-1}} T_z = T_1 = 1$, but $T_z T_{z^{-1}} \neq 1$.