Mathematics 730
Fundamental Concepts of Topology
Homework Due Wednesday, October 21, 1998
Smooth Manifolds and Maps

For this problem, think of the 2-sphere S^2 as a the one-point compactification $\mathbb{C} \cup \{\infty\}$ of the complex plane.

1. Show that S^2 has a structure of smooth 2-manifold, determined by the charts $\phi_0 = \mathrm{id} : \mathbb{C} \to \mathbb{C} \cong \mathbb{R}^2$ and $\phi_\infty = (\mathbb{C} \setminus \{0\}) \cup \infty \to \mathbb{C} \cong \mathbb{R}^2$ given by $\phi_\infty(z) = \frac{1}{z}$. (Here we are using complex arithmetic and, by definition, $\frac{1}{\infty} = 0$.)

2. Let f be a meromorphic function defined on the complex plane, in the sense of complex analysis. (That means that f is an analytic complex-valued function on $\mathbb{C} \setminus D$, where D is a discrete set of poles.) Show that if we define f to have the value ∞ at the set of poles D, then f becomes a smooth map $\mathbb{C} \cong \mathbb{R}^2 \to S^2$ (with respect to the smooth structure defined in (1)). Compute (in terms of the complex derivative f') the differential of f as a map $\mathbb{R}^2 \to \mathbb{R}^2$, both at points where f is analytic and at the poles. What are the critical points and critical values?

3. The Weierstrass function $\wp$ is defined in complex analysis to be a doubly periodic meromorphic function with a single double pole in each fundamental domain. For present purposes, let the period lattice Λ be $\mathbb{Z} + i\mathbb{Z}$ (which is identified with $\mathbb{Z}^2$ when we identify $\mathbb{C}$ with $\mathbb{R}^2$ the usual way). Then, by definition,

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

Show using (2) that $\wp$ induces a smooth map $\mathbb{R}^2/\mathbb{Z}^2 \cong T^2 \to S^2$. Describe, as precisely as possible, the critical points and critical values of this map, and the nature of the inverse images $\wp^{-1}(w)$ both when w is a regular value and when w is a critical value of the map.