MATH 740, Fall 2012
Riemannian Geometry
Homework Assignment #1:
Manifolds and Tangent Vectors

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due Monday, September 10, 2012

1. Suppose $D\colon C^\infty(\mathbb{R}) \to \mathbb{R}$ is linear and satisfies the Leibniz product rule for directional derivatives at 0:
$$D(fg) = D(f)g(0) + f(0)D(g) \tag{1}$$
for all f, $g \in C^\infty(\mathbb{R})$.
   (a) Show that (1) implies $D(1) = 0$. (Here 1 means the constant function with value 1 everywhere.)
   (b) Let $c = D(x)$, where x is the usual coordinate on $\mathbb{R}$. Show that $D(f) = cf'(0)$ for any $f \in C^\infty(\mathbb{R})$. Hint: Because of (a), it's enough to consider functions f with $f(0) = 0$. For such functions,
$$h(x) = \begin{cases} f'(0), & x = 0, \\ \frac{f(x)}{x}, & x \neq 0, \end{cases}$$
is continuous and differentiable, even C^∞, because of L'Hôpital's Rule. So $f(x) = x\,h(x)$ and $D(f) = ch(0) = cf'(0)$ by (1).
   (c) Deduce that D is automatically *local* at 0, i.e., that $D(f) = 0$ if f vanishes in some neighborhood of 0. Also deduce that D is automatically *continuous* in the C^∞ topology, that is, if f_j and f are all supported in the same neighborhood of 0, and if $f_j^{(n)} \to f^{(n)}$ uniformly for all n, then $D(f_j) \to D(f)$. (Actually you only need this for $n = 1$.)

2. Generalize problem 1 to n dimensions. In other words, if $D\colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is linear and satisfies (1) for all f, $g \in C^\infty(\mathbb{R}^n)$, show that
$$D(f) = \sum_{j=1}^{n} c_j \frac{\partial f}{\partial x_j}(0) \qquad \text{for all } f \in C^\infty(\mathbb{R}^n), \tag{2}$$

where the constants c_j can be computed as $c_j = D(x_j)$. You'll need the fact (which can be deduced from the multi-variable Taylor's Theorem) that if $f(0) = 0$, then $f(x) = \sum_{j=1}^{n} x_j h_j(x)$, where $h_j(0) = \frac{\partial f}{\partial x_j}(0)$.

3. Deduce from problem 2 and reduction to coordinate patches that it makes sense on a smooth manifold M^n to define the tangent space $T_x M$ to M at x as we did in class, and that this gives an n-dimensional vector space at each point.

4. The *Riemann sphere* $\mathbb{CP}^1$ is the one-point compactification of the complex plane $\mathbb{C}$. In other words, as a set it is $\mathbb{C} \cup \{\infty\}$, and if $z_j \in \mathbb{C}$, then $z_j \to \infty$ if $|z_j| \to \infty$. We make $\mathbb{CP}^1$ into a smooth manifold by covering it with two coordinate charts: $\mathbb{C}$ with the identity map, and $\mathbb{CP}^1 \smallsetminus \{0\} = (\mathbb{C} \smallsetminus \{0\}) \cup \{\infty\}$ with the map $\mathbb{CP}^1 \smallsetminus \{0\} \to \mathbb{C}$ defined by $\infty \mapsto 0$ and $z \mapsto \frac{1}{z}$ for $z \in \mathbb{C}$, $z \neq 0$. Show that this atlas makes $\mathbb{CP}^1$ into a smooth manifold, and that the map $z \mapsto z^n$, $n \in \mathbb{Z}$, extends to a smooth map $\mathbb{CP}^1 \to \mathbb{CP}^1$ sending ∞ to ∞ if $n > 0$, to 0 if $n < 0$, and to 1 if $n = 0$.