MATH 740, Fall 2012
Riemannian Geometry
Homework Assignment #2:
Lie Groups and Riemannian Metrics

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due Friday, September 21, 2012

1. Let A be an $n \times n$ real matrix. Show that if $A^t A = I$ and $v, w \in \mathbb{R}^n$ (viewed as column vectors), then $\langle Av, Aw\rangle = \langle v, w\rangle$, i.e., A preserves the usual inner product $\langle v, w\rangle = \sum_{i=1}^n v_i w_i$ on $\mathbb{R}^n$. A matrix satisyfing this condition is called *orthogonal*.

   (a) Show that the orthogonal matrices $O(n)$ are a closed bounded set in $M_n(\mathbb{R})$ (the $n \times n$ matrices, which we can identify with $\mathbb{R}^{n^2}$) and are thus compact. Also show that they are a group under multiplication.

   (b) Show that $O(n)$ is a smooth manifold, by checking that I is a regular value of the map $A \mapsto A^t A$ from $M_n(\mathbb{R})$ to the subspace of symmetric matrices.

   (c) Show that for any $n \geq 1$, $O(n)$ has precisely two connected components, and that the one containing the identity matrix is $SO(n)$, the *special orthogonal group* of orthogonal matrices with determinant 1. Show also that $SO(2)$ can be identified with S^1.

   (d) Show that the tangent space to $O(n)$ at the identity matrix can be identified with $\mathfrak{g} = \{X \in M_n(\mathbb{R}) : X^t = -X\}$, and that if $X \in \mathfrak{g}$, $\varphi\colon\ t \mapsto e^{tX}$ (matrix exponential) is a curve in $O(n)$ with $\varphi'(0) = X$.

   (e) With $X \in \mathfrak{g}$ and φ as in (d), show that $(Xf)(A) = \frac{d}{dt}\big|_{t=0} f(A\varphi(t))$ is the corresponding left-invariant vector field. Also show that the bracket $[X, Y]$ of two such vector fields is the left-invariant vector field corresponding to the matrix commutator $[X, Y] = XY - YX$.

2. Do problem #7 in do Carmo, Ch. 1, pp. 46–47, showing that every compact Lie group has a bi-invariant Riemannian metric.