MATH 740, Fall 2012
Riemannian Geometry
Homework Assignment #3:
Riemannian Metrics and Connections

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due Friday, September 28, 2012

1. (a) Let g be a Riemannian metric on the open disk $\overset{\circ}{D}{}^n(r)$ of radius r around 0 in Euclidean n-space $\mathbb{R}^n$. Let x be a point in $\overset{\circ}{D}{}^n(r)$, say with $|x| = \varepsilon < r$, and let λ be the minimal eigenvalue of $g(x)$ for $|x| \leq \varepsilon$. (Here we think of g as a smooth map from $\overset{\circ}{D}{}^n(r)$ to positive-definite $n \times n$ matrices. The minimum exists by a simple compactness argument.) Show that the Riemannian distance from 0 to x is *bounded below* by $\sqrt{\lambda}\,\varepsilon$. In particular it's not zero. The idea is to get a lower bound on $\int_0^a g(\gamma'(t), \gamma'(t))^{1/2}\, dt$ for any curve $\gamma\colon [0, a] \to \overset{\circ}{D}{}^n(r)$ with $|\gamma(a)| = \varepsilon$.

   (b) Deduce from (a) that the Riemannian distance between distinct points x and y on any connected Riemannian manifold cannot be 0. (Hint: You can assume y corresponds to 0 in some coordinate patch.)

2. Let ∇ be an affine connection on $\mathbb{R}$.

   (a) Show that ∇ is determined by a single smooth function $\Gamma = \Gamma^1_{1,1}$, where $\nabla_{\partial/\partial x}\left(\frac{\partial}{\partial x}\right) = \Gamma^1_{1,1}(x)\,\frac{\partial}{\partial x}$. Then show that ∇ is automatically torsion-free; in other words, that for any vector fields X and Y, $\nabla_X Y - \nabla_Y X = [X, Y]$. Hint: Write X and Y in terms of $\frac{\partial}{\partial x}$.

   (b) In terms of Γ, what is the parallel transport of the tangent vector $\frac{\partial}{\partial x}$ at 0 to a point x? Why does it only depend on x and not on the curve used to get there?