MATH 740, Fall 2012
Riemannian Geometry
Homework Assignment #4:
Geodesics and Riemannian Curvature

Jonathan Rosenberg

due Friday, October 12, 2012

(See Do Carmo Exercise #4 in Ch. 1, Exercise #8 in Ch. 2, and Example 3.10 in Ch. 3 for hints.) Let H be the *hyperbolic plane*, the upper half-plane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ with metric $g(x, y) = \frac{1}{y^2} I$ (where I is the 2×2 identity matrix, which gives the Euclidean metric on $\mathbb{R}^2$).

1. Show that horizontal translations $(x, y) \mapsto (x + t, y)$ are isometries of H.

2. Show that *inversion* $\tau\colon (x, y) \mapsto \frac{1}{x^2+y^2}\big(-x, y\big)$ is also an isometry of H. (In terms of $z = x+iy$, inversion is the map $z \mapsto \frac{-1}{z}$.) Hint: Compute $D\tau$ and show that
$$\langle (D_{(x,y)}\tau)(v), (D_{(x,y)}\tau)(w)\rangle_{\tau(x,y)} = \langle v, w\rangle_{(x,y)}.$$

3. Deduce that the isometry group of H acts *transitively* on H and that the geodesics in this metric are the vertical straight lines as well as semi-circles meeting the x-axis orthogonally. (Note that the x-axis isn't part of H, and in fact the Riemannian distance from any point in H to the x-axis is infinite, since $\frac{1}{y}$ is not integrable near $y = 0$ along vertical lines.)

4. Show that with respect to the coordinates $x_1 = x$, $x_2 = y$ on H, the Christoffel symbols of the Riemannian connection are given by $\Gamma^1_{1,1} = \Gamma^2_{1,2} = \Gamma^2_{2,1} = \Gamma^1_{2,2} = 0$, $\Gamma^2_{1,1} = -\Gamma^1_{1,2} = -\Gamma^1_{2,1} = -\Gamma^2_{2,2} = \frac{1}{y}$.

5. Finally, compute $R\big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\big)$, where R is the curvature tensor.