MATH 740, Fall 2012
Riemannian Geometry
Homework Assignment #6:
Submanifolds and the Second Fundamental Form

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Partial Solutions

2. Recall that we showed in class that if $\overline{M} = \mathbb{R}^{n+1}$ and M is the graph of a smooth function f of n variables, then at a point $x \in \mathbb{R}^n$ where $(\nabla f)(x) = 0$ (so that the tangent plane $T_{(x,f(x))}M$ is horizontal), the second fundamental form B is $(\operatorname{Hess} f)(x)$ times the unit normal vector $(0, \cdots, 0, 1)$. Deduce from this (by rotating to this situation) that if $n = 2$ and $\sigma\colon U \to \mathbb{R}^3$ is a smooth embedding, where U is an open subset of $\mathbb{R}^2$, then the second fundamental form in the direction of a unit normal vector field η (with respect to the frame transported from $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$) is given by $(\operatorname{Hess} \sigma) \cdot \eta$.

   Solution. I apologize for the confusing way this was phrased. The meaning is this. σ is a local coordinate patch on M, a surface embedded in $\mathbb{R}^3$. We give M the metric induced from this embedding, and let ∇ denote the covariant derivative or gradient (depending on context) on Euclidean space. The vector fields $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ on U go over to a local frame X, Y on M via $d\sigma$. (The problem was phrased incorrectly, however, as X and Y may not be orthonormal.) We want to compute $B(X, X)$, $B(X, Y)$, and $B(Y, Y)$ as multiples of the unit normal vector field η (which in turn can be computed by taking $X \times Y$, the vector cross product of the frame, and dividing by $|X \times Y|$ to get a unit vector field. When $\sigma(x, y) = (x, y, f(x, y))$ and at a point $\sigma(x, y)$ with $\nabla f(x, y) = 0$, η is at this point equal to $\frac{\partial}{\partial x_3}$ and we know $B = (\operatorname{Hess} f)\eta$. The claim in general is that

$$\langle B(X, X), \eta\rangle = \left\langle \frac{\partial^2 \sigma}{\partial x^2}, \eta \right\rangle,$$
$$\langle B(Y, Y), \eta\rangle = \left\langle \frac{\partial^2 \sigma}{\partial y^2}, \eta \right\rangle,$$
$$\langle B(X, Y), \eta\rangle = \left\langle \frac{\partial^2 \sigma}{\partial x \partial y}, \eta \right\rangle.$$

These are all quite similar so we just prove the first one. We know that

$$\langle B(X,X),\eta\rangle = \langle S_\eta(X),X\rangle, \quad \text{while } S_\eta(X) = -\nabla_X(\eta)^T.$$

Since X is tangential, so that inner product with X ignores the normal component, we get

$$\langle B(X,X),\eta\rangle = -\langle \nabla_X(\eta),X\rangle.$$

But $X = \frac{\partial\sigma}{\partial x}$, so

$$0 = X\langle \eta, X\rangle = \langle \nabla_X(\eta),X\rangle + \langle \eta,\nabla_X(X)\rangle = -\langle B(X,X),\eta\rangle + \left\langle \eta, \frac{\partial^2\sigma}{\partial x^2}\right\rangle,$$

and the result follows. □

3. The *Scherk minimal surface* is given by the equation $e^z = \frac{\cos x}{\cos y}$ in $\mathbb{R}^3$. Compute its second fundamental form at a point (x,y,z) on the surface (use problem 2), and show that the mean curvature (the trace of B) vanishes. Also compute the Gaussian curvature (the product of the two principal curvatures) as a function of x and y.

Solution. Let $\sigma(x,y) = \big(x,y,\ln(\cos x) - \ln(\cos y)\big)$ for $|x|,|y| < \frac{\pi}{2}$. In the above notation,

$$\begin{aligned} X &= \frac{\partial\sigma}{\partial x} = \frac{\partial}{\partial x_1} - (\tan x)\frac{\partial}{\partial x_3}, \quad Y = \frac{\partial\sigma}{\partial y} = \frac{\partial}{\partial x_2} + (\tan y)\frac{\partial}{\partial x_3},\\ \eta &= \frac{1}{\sqrt{1+\cos^2 x\tan^2 y}}\left(\sin x\frac{\partial}{\partial x_1} - \cos x\tan y\frac{\partial}{\partial x_2} + \cos x\frac{\partial}{\partial x_3}\right). \end{aligned}$$

Now we have

$$\begin{aligned} \langle B(X,X),\eta\rangle &= \frac{1}{\sqrt{1+\cos^2 x\tan^2 y}}\left\langle \frac{\partial^2\sigma}{\partial x^2}, \sin x\frac{\partial}{\partial x_1} - \cos x\tan y\frac{\partial}{\partial x_2} + \cos x\frac{\partial}{\partial x_3}\right\rangle\\ &= \frac{1}{\sqrt{1+\cos^2 x\tan^2 y}}\left\langle -(\sec^2 x)\frac{\partial}{\partial x_3}, \cos x\frac{\partial}{\partial x_3}\right\rangle\\ &= \frac{-\sec x}{\sqrt{1+\cos^2 x\tan^2 y}}. \end{aligned}$$

Similarly

$$\begin{aligned} \langle B(Y,Y),\eta\rangle &= \frac{1}{\sqrt{1+\cos^2 x\tan^2 y}}\left\langle \frac{\partial^2\sigma}{\partial y^2}, \sin x\frac{\partial}{\partial x_1} - \cos x\tan y\frac{\partial}{\partial x_2} + \cos x\frac{\partial}{\partial x_3}\right\rangle\\ &= \frac{1}{\sqrt{1+\cos^2 x\tan^2 y}}\left\langle (\sec^2 y)\frac{\partial}{\partial x_3}, \cos x\frac{\partial}{\partial x_3}\right\rangle\\ &= \frac{\cos x\sec^2 y}{\sqrt{1+\cos^2 x\tan^2 y}} \end{aligned}$$

and $B(X,Y)=0$ since $\frac{\partial^2\sigma}{\partial x\partial y}=0$.

Now X and Y are not orthonormal; we have

$$|X|=\sqrt{1+\tan^2 x}=\sec x,\quad |Y|=\sqrt{1+\tan^2 y}=\sec y.$$

We can get an orthonormal frame on M by taking

$$X'=\frac{X}{|X|}=\frac{1}{\sec x}\left(\frac{\partial}{\partial x_1}-(\tan x)\frac{\partial}{\partial x_3}\right)=(\cos x)\frac{\partial}{\partial x_1}-(\sin x)\frac{\partial}{\partial x_3}$$

and then applying Gram-Schmidt. First take $Y''=Y-\langle Y,X'\rangle X'$ so that $Y''\perp X'$, then normalize to get

$$Y'=\frac{1}{\sqrt{1+\cos^2 x\tan^2 y}}\left(\cos x\sin x\tan y\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}+\cos^2 x\tan y\frac{\partial}{\partial x_3}\right).$$

In this basis we have $B(X',X')=(\cos^2 x)B(X,X)$ and so

$$\langle B(X',X'),\eta\rangle=\frac{-\cos x}{\sqrt{1+\cos^2 x\tan^2 y}}.$$

Similarly

$$\begin{aligned}\langle B(Y'',Y''),\eta\rangle&=\langle B(Y,Y),\eta\rangle+\sin^2 x\tan^2 y\langle B(X',X'),\eta\rangle\\&=\frac{\cos x\sec^2 y}{\sqrt{1+\cos^2 x\tan^2 y}}(1-\sin^2 x\sin^2 y).\end{aligned}$$

So

$$\begin{aligned}\langle B(Y',Y'),\eta\rangle&=\frac{1}{1+\cos^2 x\tan^2 y}\langle B(Y'',Y''),\eta\rangle\\&=\frac{\cos x\sec^2 y}{(1+\cos^2 x\tan^2 y)^{3/2}}(1-\sin^2 x\sin^2 y).\end{aligned}$$

The mean curvature is therefore

$$\begin{aligned}&\frac{-\cos x}{\sqrt{1+\cos^2 x\tan^2 y}}+\frac{\cos x\sec^2 y}{(1+\cos^2 x\tan^2 y)^{3/2}}(1-\sin^2 x\sin^2 y)\\&=\frac{(-\cos x)(1+\cos^2 x\tan^2 y)+\cos x\sec^2 y(1-\sin^2 x\sin^2 y)}{(1+\cos^2 x\tan^2 y)^{3/2}}\\&=\frac{\cos x\tan^2 y-\cos^3 x\tan^2 y-\cos x\sin^2 x\tan^2 y}{(1+\cos^2 x\tan^2 y)^{3/2}}\\&=(\cos x\tan^2 y)\frac{1-\cos^2 x-\sin^2 x}{(1+\cos^2 x\tan^2 y)^{3/2}}=0.\end{aligned}$$

□