MATH 740, Fall 2012
Riemannian Geometry
Homework Assignment #7:
Noncompact Manifolds and Completeness

Jonathan Rosenberg

due Friday, November 16, 2012

1. Consider the Riemannian metric g on $\mathbb{R}$ which is $\frac{1}{\lambda(x)}$ times the usual metric, where λ is a positive smooth function.

   (a) Compute the Riemannian distance from 0 to x (where $x \in \mathbb{R}$ is either positive or negative) and show that the metric completion of $(\mathbb{R}, g)$ can be any of $(-\infty, \infty)$, $(-\infty, \infty]$, $[-\infty, \infty)$, or $[-\infty, \infty]$.

   (b) Show that only in the first case is $\mathbb{R}$ complete, and derive a necessary and sufficient condition for this in terms of the conformal factor λ. Also show that $(\mathbb{R}, g)$ is extendable in the other cases.

   (c) Show that in all cases, $(\mathbb{R}, g)$ has the property that any two points can be joined by a length-minimizing geodesic segment, so that this property is not equivalent to completeness.

2. Do problem #4 in Do Carmo, Ch. 7, showing that the universal cover of $\mathbb{R}^2 \smallsetminus \{(0,0)\}$, with the pull-back of the Euclidean metric on $\mathbb{R}^2$ is not complete but also not extendable. Thus completeness is not equivalent to non-extendability.

3. Do problem #12 in Do Carmo, Ch. 7, showing that if M is a connected Riemannian manifold with the property that for all $p, q \in M$, there is an isometry of M taking p to q, then M is necessarily complete.