MATH 740, Fall 2012
Homework Assignment #8:
Spaces of Constant Curvature, Comparison Theorems
Solutions

Jonathan Rosenberg

due Wednesday, December 12, 2012

1. A connected Riemannian manifold M is called a *Riemannian symmetric space* if, for all $x \in M$, there exists an isometry s_x of M fixing x and with $ds_x = -\mathrm{id}$. (Thus s_x reverses the direction of all geodesics starting at x.)

   (a) Prove that a Riemannian symmetric space is complete. (Show that all geodesics are infinitely extendable.)

   Solution. We show that M is geodesically complete. Let γ be a geodesic in M with $\gamma(0) = x$, $\gamma'(0)$ a unit vector in T_xM. Suppose γ is defined on an open interval containing $[0, a]$. Let $y = \gamma(a)$; then ds_y sends $\gamma'(a)$ to $-\gamma'(a)$, so that if we define $\gamma(a+t) = s_y(\gamma(a-t))$ for $0 \le t \le a$, this extends the old definition of γ near $t = a$ and allows γ to be defined on $[0, 2a]$. Continuing by induction, we get γ defined for all time. $\square$

   (b) Prove that if M is a Riemannian symmetric space, then the group of isometries G of M acts transitively on M, so that M can be identified with G/H for some Lie groups $G \supset H$.

   Solution. By (a) and the Hopf-Rinow Theorem, any two points x and y in M can be joined by a geodesic γ, say with $\gamma(0) = x$ and $\gamma(a) = y$. Let $z = \gamma(a/2)$. Then s_y sends x to y. $\square$

   (c) Show that with respect to the notation of (b), H is fixed by an automorphism σ of G of period 2, and that the connected component of the identity in H is exactly the connected component of the identity in G^σ. (Hint: Let σ be conjugation by the symmetry s_x at $x = eH \in G/H$.)

   Solution. Let G be the isometry group of M, which is automatically a Lie group. Fix a basepoint $x \in M$ and let H be its stabilizer. This is a closed subgroup of G (since fixing a point is a closed condition). Let σ be the inner automorphism defined by s_x.

Since s_x has order 2, σ is an involution, and since $s_x \in H$, σ maps H to itself. But more is true. We have a representation $H \to O(T_xM)$ defined by $\varphi \mapsto (d\varphi)_x$, and it's faithful (injective) since if $(d\varphi)_x = \mathrm{id}$, then φ fixes all geodesics starting at x, and thus fixes all of M (since every point in M can be reached by such a geodesic). Under this representation $H \to O(T_xM)$, s_x goes to -1, and thus σ goes to conjugation by -1, which is the identity. Thus σ is the identity on H and $H \subseteq G^\sigma$.

Now we need to show that the connected component of the identity in G^σ lies in H. For this it's enough to show that if $\varphi \in G^\sigma$ and φ is close to the identity, then $\varphi \in H$. Choose such a φ and let $\varphi(x) = y$. Since φ is close to the identity, y is close to x; we need to show $y = x$. Since y is close to x, we can assume $y = \gamma(a)$ for some small $a \geq 0$ and γ a unit-speed geodesic with $\gamma(0) = x$. The fact that $\varphi \in G^\sigma$ means (by definition of σ) that φ commutes with s_x. So

$$\gamma(a) = y = \varphi(x) = s_x \circ \varphi \circ s_x(x) = s_x(\varphi(x)) = s_x(\gamma(a)) = \gamma(-a).$$

For small a, the exponential map is a diffeomorphism on the ball of radius a, so this forces $a = 0$ and $y = x$. $\square$

(d) Show that all complete simply connected manifolds of constant curvature are Riemannian symmetric spaces. Find G and H in each of the cases $\mathbb{R}^n$, S^n, and H^n.

Solution. Checking the hypothesis for each of these spaces is easy. Since the isometry groups act transitively on $\mathbb{R}^n$, H^n and S^n, it's enough to construct a symmetry at one basepoint in each. For $\mathbb{R}^n$, or for H^n in the "ball realization" (see Do Carmo, p. 177), take the basepoint to be 0 and let s be multiplication by -1. For $S^n \subset \mathbb{R}^{n+1}$, take the basepoint to be the north pole $(0, \cdots, 0, 1)$ and let $s(x_1, \cdots x_n, x_{n+1}) = (-x_1, \cdots - x_n, x_{n+1})$. The corresponding homogeneous spaces are $\mathbb{R}^n = (\mathbb{R}^n \rtimes O(n))/O(n)$ and (for $n \geq 2$) $S^n = O(n+1)/O(n)$, $H^n = O(n,1)_+/O(n)$. (Note: The group $O(n,1)$ of isometries of Lorentz space $\mathbb{R}^{n,1}$ actually has 4 components. Since H^n can be identified with a single sheet of a two-sheeted hyperboloid in $\mathbb{R}^{n,1}$, we need to cut down to the subgroup $O(n,1)_+$ of index 2 that sends this sheet back to itself rather than to the opposite sheet. This subgroup still has two components, since there are orientation-reversing isometries of the hyperboloid.) $\square$

2. Let M^n be a Riemannian manifold, $x \in M$. Compute the first two terms in the series expansion of $\mathrm{vol} B_r(x)$ as a function of r. You should find that the leading term only depends on the dimension n, not the metric, and that the next term after that involves R_x, the scalar curvature at x. If the Ricci curvature controls the growth of volumes of balls, why does it not appear in these two terms?

Solution. The result of this problem can be found in the paper by Alfred Gray, "The volume of a small geodesic ball of a Riemannian manifold," Michigan Math. J. **20** (1973), 329–344.

Recall that we had the formula

$$\operatorname{vol} B_r(x) = \int_S \int_0^{\min(r,\operatorname{cut}(v))} (\det J(t))\, dt\, dv,$$

where v runs over S the unit ball S in T_xM and $\operatorname{cut}(v)$ is the distance out to the cut locus, and J is an $(n-1)\times(n-1)$ matrix of Jacobi fields J_i along the geodesic γ with $\gamma(0)=x$, $\gamma'(0)=v$. Here $J_i(0)=0$ and the $J_i'(0)=e_i$, $i=1,\cdots,n-1$, are an orthonormal basis for $v^\perp$. When we compute the series expansion for small r, we can ignore the cut locus and use the series expansion of $J_i(t)$. The Jacobi equation $J_i''+R(\gamma',J_i)\gamma'=0$ after differentiating gives $J_i'''(0)+R(v,e_i)v=0$ or $J_i'''(0)=-R(v,e_i)v$, so the Taylor series of J_i is: $J_i(t)\approx te_i-\frac{1}{6}R(v,e_i)vt^3+\cdots$. Thus the leading term in our expansion is just

$$\operatorname{vol} B_r(x) \approx \int_S \int_0^r t^{n-1}\, dt\, dv = \operatorname{vol}(S)\frac{r^n}{n} = \operatorname{vol} \overline{B}_r(0),$$

where $\overline{B}_r(0)$ is the ball of radius r in Euclidean space. Since $\det(tI_{n-1}+t^3A+\cdots)\approx t^{n-1}+t^{n+1}\operatorname{tr} A$, $\det J\approx t^{n-1}-\frac{1}{6}t^{n+1}\operatorname{Ric}_x(v)$, and the next term of the expansion of $B_r(x)$ is

$$\begin{aligned} -\frac{1}{6}\int_S\int_0^r t^{n+1}\operatorname{Ric}_x(v)\, dt\, dv &= -\frac{1}{6}\left(\int_0^r t^{n+1}\, dt\right)\left(\int_S \operatorname{Ric}_x(v)\, dv\right) \\ &= -\frac{1}{6}\operatorname{vol}(S)\frac{r^{n+2}}{n+2}\frac{R_x}{n} = -\frac{r^2}{6(n+2)}R_x \operatorname{vol}\overline{B}_r(0). \end{aligned}$$

Here we've used the result of Do Carmo, Ch. 4, Exercise 9. Thus the expansion starts with $\operatorname{vol}\overline{B}_r(0)\left(1-\frac{r^2}{6(n+2)}R_x+\cdots\right)$. Only the scalar curvature appears since we integrate the Ricci curvature over the sphere and thus the Ricci tensor is "averaged out." But when we look at volumes of larger balls, the asymmetry of the Ricci tensor shows up. □

3. Do problem 4 in Chapter 9 of Do Carmo, that any closed geodesic in an orientable complete manifold of even dimension and positive sectional curvature is homotopic to a curve of shorter length. (This provides a slight variant on the proof of this part of Synge's Theorem.)

   Solution. Let M be an orientable complete manifold of even dimension and let γ be a unit-speed geodesic in M with $\gamma(0)=\gamma(a)=x$ for some $a>0$, $\gamma'(0)=\gamma'(a)$. (Thus γ at time t comes back to where it started with the same tangent vector.) Let $\varphi\colon T_xM\to T_xM$ be the result of parallel transport along γ for $0\le t\le a$. Then φ is an orthogonal endomorphism of T_xM (since parallel transport preserves length) and sends $\gamma'(0)$ to itself. Since M is orientable, $\det\varphi=+1$. Apply Lemma 3.8 in Chapter 9 of Do Carmo to the orthogonal complement of $\gamma'(0)$, and we see that φ fixes some unit vector $v\perp\gamma'(0)$. Apply parallel transport of v along γ to get a parallel unit-speed vector field V along γ. Then apply the second variation formula to see that the second derivative of the energy for the variation of gamma defined by V is negative. This means the length of γ can be shortened by a small homotopy. □

4. Show by example that the volume comparison of the Bishop-Gromov Theorem only works one way (to show $\operatorname{vol} B_r(p) \leq \operatorname{vol} \overline{B}_r$ in the presence of a lower Ricci bound). You cannot deduce $\operatorname{vol} B_r(p) \geq \operatorname{vol} \overline{B}_r$ from an upper Ricci bound, or even an upper sectional curvature bound. (Hint: the problem shows up even when M has constant curvature but is not simply connected. Show that for fixed r, $\operatorname{vol} B_r(p)$ can be arbitrarily small if you vary the manifold M within the class of flat manifolds, those with constant curvature 0.)

   Solution. Fix $\lambda > 0$ and let $M = \mathbb{R}^n/(\lambda\mathbb{Z})^n$ with the quotient metric from the flat metric on $\mathbb{R}^n$. Then M is a flat torus of volume λ^n. So for any $p \in M$ and any $r > 0$, $\operatorname{vol} B_r(p) \leq \lambda^n$. Since λ can be arbitrarily small, this shows curvature information alone does not suffice to give a lower bound for volumes of balls. $\square$