MATH 740, Fall 2012
Riemannian Geometry
Homework Assignment #8:
Spaces of Constant Curvature, Comparison Theorems

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due Wednesday, December 12, 2012

1. A connected Riemannian manifold M is called a *Riemannian symmetric space* if, for all $x \in M$, there exists an isometry s_x of M fixing x and with $ds_x = -\mathrm{id}$. (Thus s_x reverses the direction of all geodesics starting at x.)

   (a) Prove that a Riemannian symmetric space is complete. (Show that all geodesics are infinitely extendable.)

   (b) Prove that if M is a Riemannian symmetric space, then the group of isometries G of M acts transitively on M, so that M can be identified with G/H for some Lie groups $G \supset H$.

   (c) Show that with respect to the notation of (b), H is fixed by an automorphism σ of G of period 2, and that the connected component of the identity in H is exactly the connected component of the identity in G^σ. (Hint: Let σ be conjugation by the symmetry s_x at $x = eH \in G/H$.)

   (d) Show that all complete simply connected manifolds of constant curvature are Riemannian symmetric spaces. Find G and H in each of the cases $\mathbb{R}^n$, S^n, and H^n.

2. Let M^n be a Riemannian manifold, $x \in M$. Compute the first two terms in the series expansion of $\mathrm{vol} B_r(x)$ as a function of r. You should find that the leading term only depends on the dimension n, not the metric, and that the next term after that involves R_x, the scalar curvature at x. If the Ricci curvature controls the growth of volumes of balls, why does it not appear in these two terms?

3. Do problem 4 in Chapter 9 of Do Carmo, that any closed geodesic in an orientable complete manifold of even dimension and positive sectional curvature is homotopic to a curve of shorter length. (This provides a slight variant on the proof of this part of Synge's Theorem.)

4. Show by example that the volume comparison of the Bishop-Gromov Theorem only works one way (to show $\operatorname{vol} B_r(p) \leq \operatorname{vol} \overline{B}_r$ in the presence of a lower Ricci bound). You cannot deduce $\operatorname{vol} B_r(p) \geq \operatorname{vol} \overline{B}_r$ from an upper Ricci bound, or even an upper sectional curvature bound. (Hint: the problem shows up even when M has constant curvature but is not simply connected. Show that for fixed r, $\operatorname{vol} B_r(p)$ can be arbitrarily small if you vary the manifold M within the class of flat manifolds, those with constant curvature 0.)