

The Based Homotopy Lifting Property

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Theorem 1 *Let $A \hookrightarrow X$ be a cofibration and let $Z = X \times \{0\} \cup_{A \times \{0\}} A \times [0, 1]$. Then $(X \times [0, 1], Z)$ is a DR (deformation retract) pair. In other words, there is a continuous map $u : X \times [0, 1] \rightarrow [0, 1]$ such that $Z = u^{-1}(0)$, and there is a map $h : X \times [0, 1] \times [0, 1] \rightarrow X \times [0, 1]$ with $h(x, t, 0) = (x, t)$, $h(x, t, 1) \in Z$ for $(x, t) \in X \times [0, 1]$, $h(x, t, s) = (x, t)$ for $(x, t) \in Z$ and $s \in I = [0, 1]$.*

Proof. We know (by a theorem proved in class) that there is a retraction $r : X \times [0, 1] \rightarrow Z$. Let π_j , $j = 1, 2$, be projection on the j -th factor in $X \times [0, 1]$. For $x \in X$, define $w(x) = \max_{t \in I} (t - \pi_2(r(x, t)))$. Clearly u is continuous and real-valued, and since $r(x, 0) = (x, 0)$, $w(x) \geq 0$. On the other hand, $t - \pi_2(r(x, t)) \leq t \leq 1$, so u is a (continuous) map $X \rightarrow [0, 1]$. If $x \in A$, then $r(x, t) = (x, t)$ for all t , so $w(x) = 0$. On the other hand, if $w(x) = 0$, then for all $0 < t \leq 1$, $t - \pi_2(r(x, t)) \leq 0$, hence $\pi_2(r(x, t)) \geq t > 0$, so $r(x, t) \notin X \times \{0\}$, hence $r(x, t) \in A \times I$. Since $(x, 0) = r(x, 0) = \lim_{t \rightarrow 0} r(x, t)$, passing to the limit as $t \rightarrow 0$ shows that $x \in A$. So $A = w^{-1}(0)$. Now define $u(x, t) = \min(w(x), t)$. This is 0 precisely when either $w(x) = 0$ or $t = 0$, or in other words, when $(x, t) \in Z$. This proves the first part.

For the second part, we need to construct a deformation retraction h . The idea is that in some sense (x, t) is closer to $A \times I$ when $w(x) < t$, and closer to $X \times \{0\}$ when $t < w(x)$. So define h by the formula

$$h(x, t, s) = \begin{cases} (\pi_1 \circ r(x, s), t - sw(x)), & w(x) \leq t, \\ (\pi_1 \circ r(x, st/w(x)), t(1 - s)), & w(x) \geq t. \end{cases}$$

This is well-defined since if $w(x) = t$, both formulas reduce to

$$(\pi_1 \circ r(x, s), t(1 - s)).$$

If $s = 0$, both formulas give $(\pi_1 \circ r(x, 0), t)$, and since $\pi_1 \circ r(x, 0) = \pi_1(x, 0) = x$, we have $h(x, t, 0) = (x, t)$, one of the requirements. If $t = 0$ so that $(x, 0) \in Z$, then $h(x, 0, s) = (\pi_1 \circ r(x, 0), 0) = (x, 0)$, and so $(x, 0)$ is stationary. If $x \in A$ so that again $(x, t) \in Z$, then $w(x) = 0$ so $h(x, t, s) = (\pi_1 \circ r(x, s), t) = (x, t)$ and again (x, t) is stationary. Finally, if $s = 1$, then if $w(x) \geq t$, $h(x, t, 1) = (\pi_1 \circ r(x, t/w(x)), 0) \in X \times \{0\} \subseteq Z$, while if $w(x) < t$, then $r(x, 1) \notin X \times \{0\}$ (since this would imply $w(x) = 1$), so $r(x, 1) \in A \times I$ and $h(x, t, 1) = (\pi_1 \circ r(x, 1), t - w(x)) \in A \times I \subseteq Z$. So h is the required deformation retraction. \square

Theorem 2 Suppose $p : E \rightarrow B$ is a fibration and (Z, C) is a DR pair, i.e., C is a closed subset of Z , $C = u^{-1}(0)$ for some continuous map $u : Z \rightarrow I$, and there is a deformation retraction $H : Z \times I \rightarrow Z$ with $H(z, 0) = z$, $H(z, 1) \in C$ for $z \in Z$, $H(c, t) = c$ for $c \in C$ and $t \in I$. Then given $f : Z \rightarrow B$ and a lift g of $f|_C$, there is a lift $h : Z \rightarrow E$ of f extending g .

Proof. Let $r : Z \rightarrow C$ be the retraction $H(_, 1)$. Let

$$\Phi(z, t) = f \circ H \left(z, \max \left(0, 1 - \frac{t}{u(z)} \right) \right).$$

(When $u(z) = 0$, $z \in C$ and we interpret this to mean $f(z)$.) Then Φ is a homotopy $Z \times I \rightarrow B$ and

$$z \mapsto \Phi(z, 0) = f \circ H(z, 1) = f \circ r(z)$$

has the lift $g \circ r$. So there is a homotopy $\Psi : Z \times I \rightarrow E$ lifting Φ with $\Psi(z, 0) = g \circ r(z)$, $z \in Z$. Let $h(z) = \Psi(z, u(z))$. Then if $z \in C$, $u(z) = 0$ and $h(z) = \Psi(z, 0) = g \circ r(z)$, while for arbitrary z ,

$$p \circ h(z) = \Psi(z, u(z)) = f \circ H(z, 0) = f(z).$$

Thus h is the desired lift of f . \square

Corollary 3 Suppose $p : E \rightarrow B$ is a fibration and $A \hookrightarrow Y$ is a cofibration. Let $R = Y \times \{0\} \cup_{A \times \{0\}} A \times I$, a subset of $Y \times I$. Assume given $h : Y \times I \rightarrow B$ with H a lift of $h|_R$. Then H extends to a lift of h .

Proof. By Theorem 1, $(Y \times I, R)$ is a DR pair. So by Theorem 2, a lift of $h|_R$ extends to a lift of h . \square

Corollary 4 (Based Homotopy Lifting) Suppose $p : E \rightarrow B$ is a fibration and $\{z_0\} \hookrightarrow Z$ is a cofibration. Suppose $e_0 \in E$, $b_0 = p(e_0) \in B$ and one is given $h : Y \times I \rightarrow B$ with $h(y_0, t) = b_0$ for all t . If $h(_, 0)$ lifts to a map $g : Z \rightarrow E$ with $g(z_0) = e_0$, then h lifts to a homotopy $H : Y \times I \rightarrow E$ with $h(y_0, t) = e_0$ for all t .

Proof. Specialize Corollary 3 to the case $A = \{z_0\}$. \square