MATHEMATICS 748H: INTRODUCTION TO HOMOTOPY THEORY
EXERCISE SET #1: FUNDAMENTALS OF COFIBRATIONS AND FIBRATIONS
DUE WEDNESDAY, SEPTEMBER 18, 2002

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1. Let X be a (Hausdorff) topological space, A a subspace.
   (i) Show that if there is a retraction r from X onto A, then A is closed in X.
   (ii) (May, problem 1, p. 46) Deduce that if the inclusion of A into X is a cofibration, then A is closed in X. Hint: Use the characterization of cofibrations, together with (i), applied not to (X, A) but to some other pair.
   (iii) Show that if A is contractible and the inclusion $A \hookrightarrow X$ is a cofibration, then there is a retraction from X onto A.
   (iv) Show that not every inclusion of compact Hausdorff spaces is a cofibration. In particular, let $X = [0, 1] \times [0, 1]$ and let A be the union of the line segments joining $(0, 1)$ and $(1/n, 1)$, n a positive integer, to the origin. (This example is in Whitehead's book.) Show that A is contractible, but that there is no retraction from X onto A. Deduce from (iii) that the inclusion of A into X is not a cofibration.

2. (Whitehead Ch. I, Exercise 8) Suppose $A \hookrightarrow X$ is a cofibration and Y is contractible.
   (i) Show every map $f : A \to Y$ has an extension $g : X \to Y$.
   (ii) Show that any two maps f_0, $f_1 : X \to Y$ which agree when restricted to A are homotopic (rel. A). Hint: First show

$$(X \times \{0, 1\}) \cup (A \times [0, 1]) \hookrightarrow X \times [0, 1]$$

   is an cofibration; then use (i).
   (iii) (an application to analysis, not in Whitehead) Suppose V is a normed vector space over $\mathbb{R}$. Deduce that every uniformly bounded map $A \to V$ has an extension to X satisfying the same uniform bound.

3. Suppose $p : X \to B$ is a fibration and Y is contractible. Show that every map $f : Y \to B$ has a lifting $g : Y \to X$.