MATHEMATICS 748H: INTRODUCTION TO HOMOTOPY THEORY
EXERCISE SET #2: GROUPS OF HOMOTOPY CLASSES
DUE WEDNESDAY, SEPTEMBER 25, 2002

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1. Let (X, x_0) be a based space (always assumed compactly generated and Hausdorff). Prove that ΩX, equipped with the multiplication $\Omega X \times \Omega X \xrightarrow{\mu} \Omega X$ and inversion $\Omega X \xrightarrow{\iota} \Omega X$,

$$\begin{cases} \mu(\alpha,\beta)(s) = \begin{cases} \alpha(2s), & 0 \le s \le \frac{1}{2}. \\ \beta(2s-1), & \frac{1}{2} \le s \le 1, \end{cases} \\ \iota(\alpha)(s) = \alpha(1-s), \end{cases}$$

is an H-group. (Thus for any based space Y, $[Y, \Omega X]$ is a group.)

2. Let (X, x_0) be a based space. Recall that the (based) suspension ΣX is the smash product $X \wedge S^1$, i.e., the quotient space of the product $X \times S^1$ in which $x_0 \times S^1$ and $X \times *$ ($*$ the basepoint of S^1) are collapsed to a common point which serves as the basepoint of ΣX. In turn one can represent S^1 as the quotient group $\mathbb{R}/\mathbb{Z}$ or the quotient space $[0,1]/(0 \sim 1)$. Represent points in ΣX by equivalence classes $[x,t]$ of pairs (x,t), with $x \in X$, $t \in S^1$. Show that ΣX, equipped with the comultiplication $\nu\colon \Sigma X \to \Sigma X \vee \Sigma X$ and inversion $\Sigma X \xrightarrow{\iota} \Sigma X$,

$$\begin{cases} \nu([x,t]) = \begin{cases} [x,2t]_1, & 0 \le t \le \frac{1}{2}. \\ [x,2t-1]_2, & \frac{1}{2} \le t \le 1, \end{cases} \\ \quad \text{where the subscripts indicate which copy of } \Sigma X, \\ \iota([x,t]) = [x, 1-t], \end{cases}$$

is an H-cogroup. (Thus for any based space Y, $[\Sigma X, Y]$ is a group.) **Caution:** You need to check that ν and ι are well defined on equivalence classes and send basepoint to basepoint.

3. (Cf. May, page 56, top.) Use the fact that in the category of compactly generated spaces, $\text{Maps}(S^1 \times X, Y) \cong \text{Maps}(X, \text{Maps}(S^1, Y))$, to prove the adjunction formula $[\Sigma X, Y] \cong [X, \Omega Y]$, and show that this identification is a group isomorphism with respect to the group structures defined in #1 and #2.

4. There are several (*a priori* different) group laws defined on $[\Sigma^2 X, Y] \cong [\Sigma X, \Omega Y] \cong [X, \Omega^2 Y]$ via #1 and #2. Prove that all these group structures coincide and are abelian. (See for instance Bredon, pp. 442 ff., or Spanier, pp. 43–44.)