MATHEMATICS 748H: INTRODUCTION TO HOMOTOPY THEORY
EXERCISE SET #3: HOMOTOPY SEQUENCES
DUE WEDNESDAY, OCTOBER 2, 2002

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1. (May, top of page 59) Let $p\colon E \to B$ be a fibration, and suppose E and B are equipped with basepoints and p sends basepoint to basepoint. Show that p has the *based* homotopy lifting property for *based* maps $Y \to B$, where Y is a based space with non-degenerate basepoint y_0 (i.e., such that $\{y_0\} \hookrightarrow Y$ is a cofibration).

2. The *unitary group* is the compact group $U(n)$ of $n \times n$ complex-valued matrices u such that $u^* = u^{-1}$. (Here u^* is the adjoint with respect to the usual inner product on $\mathbb{C}^n$, i.e., the conjugate transpose of u.)

(1) Show that $U(n)$ operates transitively on the unit sphere S^{2n-1} in $\mathbb{C}^n$ (by matrix multiplication, when we think of $\mathbb{C}^n$ as consisting of column vectors), and that the isotropy group of $(0, \cdots, 0, 1)^t$ can be identified with $U(n-1)$. Here $U(n-1) \hookrightarrow U(n)$ via $u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$. Then show that the map $p\colon U(n) \to S^{2n-1}$ defined by $u \mapsto u \cdot (0, \cdots, 0, 1)^t$ is a fiber bundle (and hence a fibration) with fiber $U(n-1)$.

(2) Deduce that there is an exact sequence of homotopy groups

$$\cdots \to \pi_j(U(n-1)) \to \pi_j(U(n)) \to \pi_j(S^{2n-1}) \to \cdots$$

and thus that $\pi_j(U(n-1)) \to \pi_j(U(n))$ is an isomorphism for $j < 2n-2$.

(3) Show that the map $\pi_j(U(n-1)) \to \pi_j(U(n))$ is *not* necessarily an isomorphism for $j = 2n-1$. (Hint: what is $U(1)$?)

(4) (A little harder than (3)) Show from the same exact sequence that $\pi_j(U(n))$ contains a summand of $\mathbb{Z}$ when $n \geq (j+1)/2$ and j is odd.