Mathematics 748H: Introduction to Homotopy Theory Exercise Set #5: The Homotopy Excision and Hurewicz Theorems

Jonathan Rosenberg

due Wednesday, October 23, 2002

1. This concerns another approach to the homotopy groups of $S^n \vee S^n$.

(a). Show that the pair $(S^n \times S^n, S^n \vee S^n)$ is $(2n-1)$-connected. (Hint: $S^n \times S^n$ has a CW decomposition with 4 cells, 3 of which constitute $S^n \vee S^n$.)

(b). If X and Y are pointed spaces, show that $\pi_j(X \times Y) \cong \pi_j(X) \oplus \pi_j(Y)$. (Use the long exact sequence of the trivial fibration $X \times Y \twoheadrightarrow Y$.)

(c). Deduce from (a), (b), and the Homotopy Excision Theorem that

$$\pi_j(S^n \vee S^n) \cong \pi_j(S^n) \oplus \pi_j(S^n)$$

(with the two summands coming from the inclusion of the two spheres) in a certain range of values of j. Try to give the best possible bounds on j for which this holds, and to find explicit values of j and n for which this *fails*.

2. Let X be an $(n-1)$-connected CW complex, $n \geq 2$.

(a). Show that the natural maps

$$\pi_{n+1}(X^{(n+1)}) \to \pi_{n+1}(X), \qquad H_{n+1}(X^{(n+1)}, \mathbb{Z}) \to H_{n+1}(X, \mathbb{Z})$$

are surjective. (Look at the long exact seqeunces of the pair $(X,\, X^{(n+1)})$.)

(b). From the commutative diagram

$$\begin{array}{ccccccccc}
 & & \pi_{n+1}(X^{(n+1)}) & \longrightarrow & \pi_{n+1}(X^{(n+1)}, X^{(n)}) & \xrightarrow{\partial} & \pi_n(X^{(n)}) & \longrightarrow & \pi_n(X^{(n+1)}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_{n+1}(X^{(n+1)}) & \longrightarrow & H_{n+1}(X^{(n+1)}, X^{(n)}) & \xrightarrow{\partial} & H_n(X^{(n)}) & \longrightarrow & H_n(X^{(n+1)}) & \longrightarrow & 0,
\end{array}$$

show that the Hurewicz map $\pi_{n+1}(X^{(n+1)}) \to H_{n+1}(X^{(n+1)}, \mathbb{Z})$ is surjective. (Note that this is *one dimension higher* than the degree where the Hurewicz Theorem says one gets an isomorphism.)

(c). Deduce (from (a) and (b)) that the Hurewicz map $\pi_{n+1}(X) \to H_{n+1}(X, \mathbb{Z})$ is surjective.

(d). Give an example (again with X $(n-1)$-connected) where the Hurewicz map $\pi_{n+1}(X) \to H_{n+1}(X, \mathbb{Z})$ is non-trivial.

(e). Show that the result of (c) fails in the case $n = 1$, by considering the case where $X = T^2 = S^1 \times S^1$, the 2-torus.