Mathematics 748H: Introduction to Homotopy Theory Exercise Set #6: Eilenberg-Mac Lane Spaces and Homology and Homotopy Calculations

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1. This exercise concerns the loop space of a sphere ΩS^n, for $n \geq 2$.

(a). Show that the homology of ΩS^n, $n \geq 2$, is given by

$$\widetilde{H}_i(\Omega S^n, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & i \equiv 0 \mod (n-1),\ i > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Prove this using the fibration $\Omega S^n \to \mathcal{P}S^n \xrightarrow{p} S^n$ as follows. The total space of this fibration, the path space $\mathcal{P}S^n$, is contractible. On the other hand, the base of this fibration, S^n, is the union of two contractible hemispheres, say D^n_+ and D^n_-, intersecting in the equator S^{n-1}. So show that $p^{-1}(D^n_\pm)$ has the homotopy type of ΩS^n and that $p^{-1}(S^{n-1})$ has the homotopy type of $\Omega S^n \times S^{n-1}$. Now use the Mayer-Vietoris sequence for

$$\mathcal{P}S^n = p^{-1}(D^n_+) \cup p^{-1}(D^n_-).$$

(b). State the dual result for cohomology and derive it from the Universal Coefficient Theorem.

(c). Looking at the Mayer-Vietoris sequence in *cohomology*, observe that it translates into an isomorphism

$$H^*(p^{-1}(D^n_+)) \oplus H^*(p^{-1}(D^n_-)) \underset{\cong}{\xrightarrow{\psi}} H^*(p^{-1}(S^{n-1})) \cong H^*(\Omega S^n \times S^{n-1})$$

which is a ring homomorphism when restricted to either summand on the left. Let 1 and u be the usual generators of $H^*(S^{n-1})$ (in degrees 0 and $n-1$, respectively). Then we can think of ψ as a map

$$H^*(\Omega S^n) \oplus H^*(\Omega S^n) \to H^*(\Omega S^n) \otimes 1 \oplus H^{*-n+1}(\Omega S^n) \otimes u$$

which is a ring homomorphism when restricted to either summand on the left. Show that this map has the form

$$(x, 0) \mapsto x \otimes 1 + \theta(x) \otimes u. \tag{1}$$

(d). From multiplicativity of (1) and graded commutativity of the cohomology ring, deduce that θ is a graded derivation, i.e.,

$$\theta(xy) = \theta(x)y + (-1)^{(n-1)\deg x} x\theta(y).$$

(e). Deduce from (d) that when n is odd (so that $n-1$ is even), the canonical generator x of $H^{n-1}(\Omega S^n, \mathbb{Z})$ has the property that x^k is $k!$ times a generator of $H^{k(n-1)}(\Omega S^n, \mathbb{Z})$.

2. Now we use the results of #1 to obtain more information on homotopy groups of spheres. If you got stuck on #1, simply assume the results and continue.

Also recall that $\mathbb{CP}^\infty$ may be chosen to be a model for $K(\mathbb{Z}, 2)$, and that the cohomology ring of $\mathbb{CP}^\infty$ is the polynomial ring $\mathbb{Z}[u]$ on the canonical generator $u \in H^2(\mathbb{CP}^\infty, \mathbb{Z})$.

(a). Show that there is a map $f\colon \Omega S^3 \to \mathbb{CP}^\infty$ which is uniquely defined up to homotopy by the property that $f^*(u)$ is the canonical generator of $H^2(\Omega S^3, \mathbb{Z})$.

(b). Show from #1 and (a) that $f^*(u^2)$ is *twice* a generator of $H^4(\Omega S^3, \mathbb{Z})$.

(c). Using (b) and the relative Hurewicz theorem, deduce that $\pi_3(\Omega S^3) \cong \mathbb{Z}/2$.

(d). From (c) and the Freudenthal suspension theorem, deduce that

$$\pi_{n+1}(S^n) \cong \mathbb{Z}/2 \quad \text{for all } n \geq 3,$$

and that the generator of this group is the class of the iterated suspension of the Hopf fibration $S^3 \to S^2$.