

Mathematics 748H:
Introduction to Homotopy Theory
Exercise Set #7: Obstruction Theory,
Postnikov Systems,
and the Serre Spectral Sequence

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1. Suppose X is a CW-complex with only two non-zero homotopy groups, π_2 and π_m , where $m \geq 3$. Assume further that $\pi_2(X) \cong \mathbb{Z}$, $\pi_m(X) \cong \mathbb{Z}$. Show that if m is even, then X has the homotopy type of $K(\mathbb{Z}, 2) \times K(\mathbb{Z}, m)$, whereas if m is odd, there are infinitely many possibilities for the homotopy type of X , given by an integer k (the k -invariant!). Show that varying k really changes the homotopy type, at least when k is positive, by showing from the Serre spectral sequence that $H_m(X; \mathbb{Z}) \cong H^{m+1}(X; \mathbb{Z}) \cong \mathbb{Z}/k$. (Also compute the homology groups in dimensions less than m , and show that they are torsion-free.)
2. Show that the fact that $H^*(\mathbb{C}\mathbb{P}^\infty; \mathbb{Z})$ is a polynomial ring may be interpreted as saying that any non-zero natural transformation (“cohomology operation”) from $H^2(_; \mathbb{Z})$ to $H^k(_; \mathbb{Z})$ must be of the form $x \mapsto mx^{k/2}$ (and k must be even).
3. Apply the Serre spectral sequence to the fibration

$$K(\mathbb{Z}, 2) \rightarrow \mathcal{P}(K(\mathbb{Z}, 3)) \rightarrow K(\mathbb{Z}, 3)$$

(here the path space $\mathcal{P}(K(\mathbb{Z}, 3))$ is contractible), together with your knowledge of the cohomology ring of $K(\mathbb{Z}, 2) = \mathbb{C}\mathbb{P}^\infty$, to compute $H^j(K(\mathbb{Z}, 3); \mathbb{Z}/2)$ for $j \leq 5$. Deduce that there is a non-trivial natural transformation (“cohomology operation”)

$$H^3(_; \mathbb{Z}) \rightarrow H^5(_; \mathbb{Z}/2),$$

traditionally called Sq^2 , which cannot be expressed in terms of cup products. (Note the contrast with exercise #2!)