

Mathematics 748H:
Introduction to Homotopy Theory
Exercise Set #8: Classifying Spaces,
Vector Bundles,
and Characteristic Classes

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1. Let X be a compact space. Show that the first Chern class $c_1 \in H^2(BU(1), \mathbb{Z})$ sets up an isomorphism between the group of complex line bundles (vector bundles of rank 1) on X , with \otimes as the group operation, and the abelian group $H^2(X, \mathbb{Z})$. Show that the inverse L^{-1} of a line bundle $p: L \rightarrow X$ may be obtained by replacing each fiber of L by its dual space.
2.
 - (a). Define a map $\varphi: SU(2) \rightarrow SO(3)$ by letting $SU(2)$ act by conjugation on the 3-dimensional real vector space of 2×2 skew-hermitian ($\bar{A} = -A^t$) complex matrices with trace 0. Show that this is a 2-to-1 covering map, hence (since $SU(2) \cong S^3$ and $\varphi(g) = \varphi(-g)$, so that φ identifies antipodal points) that $SO(3) \cong \mathbb{R}P^3$.
 - (b). Use (a) (take it for granted if you got stuck on the first part) and the fibration $SO(n-1) \rightarrow SO(n) \rightarrow S^{n-1}$ to show that $\pi_1(SO(n)) \cong \mathbb{Z}$ for $n = 2$, $\mathbb{Z}/2$ for $n \geq 3$.
 - (c). Deduce that there are only two oriented real rank- n vector bundles over S^2 if $n \geq 3$, and that they are distinguished by the Stiefel-Whitney class $w_2 \in H^2(S^2, \mathbb{F}_2)$. (Remember that $\pi_2(BSO(n)) \cong \pi_1(SO(n))$.)
 - (d). Show on the other hand that there are infinitely many oriented real rank- n vector bundles over S^2 if $n = 2$, distinguished by the Euler class in $H^2(S^2, \mathbb{Z})$.
 - (e). Use (c) and (d) to show that the monoid of oriented real vector bundles over S^2 does not have the cancellation property. More precisely, if $E \rightarrow S^2$ is a rank-2 oriented real vector bundle with non-zero even Euler class, then E is non-trivial, but $E \oplus T$ is trivial for T a trivial real line bundle.

3. Recall that we showed in class that $U(2) \cong SU(2) \times U(1) \cong S^3 \times S^1$, and thus that $\pi_j(U(2)) = \pi_j(S^3)$ for $j \geq 2$. Use this fact to show that there exists a rank-2 complex vector bundle over S^5 which is not trivial, but which has trivial Chern classes. **For extra credit:** Show that this bundle is stably trivial in the sense of exercise 2(e).