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Abstract. We have discovered that our paper as listed in the title contains two mistakes. The "spin^c surgery theorem" should require that the first Chern class of the spin^c line bundle has to be primitive (i.e., not a multiple of another class in H^2), and the $\mathbb{CP}^2$-transfer map used near the end of the paper should be based on maps to $BU(3)$, not $BSU(3)$.

In the course of doing final corrections to our paper [4], which depends heavily on certain parts of our previous paper [2], we found some errors in the latter. The purpose of this Erratum is to acknowledge and correct these errors and to explain how they affect subsequent work, including not only [4] but also [3]. We felt it was important to correct the record.

There are essentially two main things in [2] that need fixing. One is the spin^c bordism theorem for what we have since come to call generalized positive scalar curvature, [2, Theorem 4.3]. This bordism theorem depends on a spin^c surgery theorem, [2, Theorem 4.2], which is correct, but the error involves the question of when one can apply the surgery theorem to deduce that one can find generalized positive scalar curvature on a simply connected non-spin spin^c manifold M^n, given that there is another spin^c manifold M' with generalized positive scalar curvature in the same spin^c bordism class. We address this issue in Section 1.

The other error is a more technical matter of homotopy theory, and involves the correct definition of one of the transfer maps used in [2, Theorem 5.1]. This is discussed in Section 2.

1. The Bordism Theorem

We begin by quoting the Bordism Theorem as stated in [2]. For convenience, we use the terminology introduced in [3], namely that for a a spin^c manifold M with spin^c line bundle L, if one is given a Riemannian metric g on M, a hermitian metric on L, and a unitary connection A on L, the *generalized scalar curvature* of (g, A) is $R^{\mathrm{gen}}_{(g,A)} := R_g - 2|\Omega|$, where R_g is the scalar curvature of g, Ω is the connection 2-form of A, and $|\Omega|$ is its pointwise operator norm (for action on spinors). Positivity of $R^{\mathrm{gen}}_{(g,A)}$ is equivalent [3, Lemma 3.1] to positivity of the *twisted scalar curvature*, which after dividing by 4 is the quantity that appears (in addition to $\nabla^*\nabla$) in the formula for D_L^2, D_L the spin^c Dirac operator of (M, L). It follows that if $\alpha^c(M, L)$ is the index of the Dirac operator D_L, then positivity of $R^{\mathrm{gen}}_{(g,A)}$ implies that $\alpha^c(M, L) = 0$. Recall that

$$\alpha^c(M, L) = \left\langle \widehat{A}(M) e^{c_1(L)/2}, [M] \right\rangle$$

is a spin^c bordism invariant of the pair (M, L). (In general, it depends on the choice of spin^c structure through the choice of the line bundle L. For simply connected manifolds, or more generally, manifolds with no 2-torsion in H^2, the choice of spin^c structure is equivalent to the choice of spin^c line bundle.)

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Theorem 1.1 ([2, Theorem 4.3])**.** *Let (M^n, L) be a simply connected spin^c manifold with $n \geq 5$. Assume that there is another spin^c manifold (M', L') which is spin^c-bordant to (M, L), and that (M', L') admits a Riemannian metric g', a hermitian metric on L', and a unitary connection A' on L' with $R^{gen}_{(g',A')}$ strictly positive. Then (M, L) also admits a Riemannian metric g, a hermitian metric on L, and a unitary connection A on L with $R^{gen}_{(g,A)}$ strictly positive.*

Unfortunately, there was a problem with the proof in [2]. The idea was to start with a spin^c bordism W from M' to M and to do spin^c surgery on it to make (W, M) 2-connected. The problem is that when doing surgery on an embedded 2-sphere in W with trivial normal bundle, the proof of the surgery theorem ([2, Theorem 4.2]) only applies if $c_1(\mathcal{L})$, $\mathcal{L}$ the spin^c line bundle of W, vanishes on the homology class of this 2-sphere. Thus one can do surgeries on W to kill $\ker c_1(\mathcal{L})$, which reduces $H_2(W)$ to $\mathbb{Z}$. If $H_2(M)$ maps **onto** this copy of $\mathbb{Z}$, all is fine and one has killed $H_2(W, M)$ as required. But in general one ends up with $H_2(W, M)$ a finite cyclic group (of odd order), which is not good enough. Thus the proof really proves the following:

Theorem 1.2 (Corrected Bordism Theorem)**.** *Let (M^n, L) be a simply connected non-spin spin^c manifold with $n \geq 5$. **Also assume that** $c_1(L)\colon H_2(M) \to \mathbb{Z}$ **is surjective, or equivalently, that the classifying map** $M \to B\mathrm{Spin}^c$ **of** (M, L) **is a 2-equivalence (an isomorphism on** π_1 **and a surjection onto** $\pi_2(B\mathrm{Spin}^c) = \mathbb{Z}$**.)** Assume that there is another spin^c manifold (M', L') which is spin^c-bordant to (M, L), and that (M', L') admits a Riemannian metric g', a hermitian metric on L', and a unitary connection A' on L' with $R^{gen}_{(g',A')}$ strictly positive. Then (M, L) also admits a Riemannian metric g, a hermitian metric on L, and a unitary connection A on L with $R^{gen}_{(g,A)}$ strictly positive.*

To see that the difference between Theorem 1.2 and (the false) Theorem 1.1 matters, we present a counterexample to Theorem 1.1 (that doesn't violate Theorem 1.2). First we need a rather easy lemma.

Lemma 1.3. *Let (M, L) be a spin^c manifold with $L = E^{\otimes k}$ $(k \geq 1)$, a tensor power of another spin^c line bundle E. Assume that (M, L) admits generalized positive scalar curvature (gpsc, i.e., there exist a Riemannian metric g on M and a hermitian metric and unitary connection A on L with $R^{gen}_{(g,A)}$ strictly positive). Then $(M, E^{\otimes j})$ admits generalized positive scalar curvature for any j with $|j| \leq k$.*

Proof. First of all, as noted in [3, §3], the condition doesn't depend on the hermitian metric on the bundle, so we can forget about that. Also, unitary connections on a line bundle form an affine space over $i\Omega^1(M)$, and $A_E \mapsto A_E^{\otimes k}$ is affine with linear part multiplication by k; hence the given connection A on L is $A_E^{\otimes k}$ for a unique unitary connection A_E on E. The curvature 2-forms are related by $\Omega_L = k\Omega_E$, so

$$R^{\mathrm{gen}}_{g,A} = R_g - 2k|\Omega_E| \leq R^{\mathrm{gen}}_{g,A_{E^{\otimes j}}} = R_g - 2|j|\,|\Omega_E|,$$

and the result follows immediately. □

Proposition 1.4. *Let Q^6 be a smooth complex quadric hypersurface in $\mathbb{CP}^4$, viewed as a smooth 6-manifold, and let $M^6 = Q^6 \# (K^4 \times S^2)$, where K is a K3 surface (viewed as a smooth 4-manifold). Then $H^\bullet(Q; \mathbb{Z}) \cong \mathbb{Z}[h, l]/(h^2 - 2l, l^2)$, h the hyperplane section class in $H^2(Q; \mathbb{Z})$, l the class in $H^4(Q; \mathbb{Z})$ of a maximal linear subspace of Q, and $w_2(Q)$ is the mod-2 reduction of h. If z is the usual generator of $H^2(S^2)$, then we can also view h and z as living in $H^2(M)$, with $hz = 0$ and $z^2 = 0$, and $ah + bz$ defines a spin^c structure if a is odd and b is even, with $\alpha^c(M, ah + bz) = \frac{a(a^2-1)}{24} + b$ (for a odd, b even).*

Proof. Since α^c is additive on connected sums, $\alpha^c(M, ah+bz) = \alpha^c(Q, ah)+\alpha^c(K\times S^2, bz)$. The structure of the cohomology ring of Q can be found in [6, Ch. 6] and in [11]. The α^c invariant of Q is effectively computed in [5]. The second summand is

$$\left\langle \widehat{A}(K\times S^2)e^{bz/2}, [M]\right\rangle = \langle \widehat{A}(K), [K]\rangle\langle bz/2, [S^2]\rangle = 2b/2 = b.$$

□

Proposition 1.5. *Let $M^6 = Q^6\#(K^4\times S^2)$ be as in Proposition 1.4 and let L be the line bundle on M with $c_1(L) = 7h - 14z$. Then M is simply connected and (M, L) is spin^c bordant to a manifold of generalized positive scalar curvature, but (M, L) does **not** admit generalized positive scalar curvature, hence is a counterexample to "Theorem" 1.1.*

Proof. That M is simply connected follows from simple connectivity of complex hypersurfaces in $\mathbb{CP}^n$, $n \geq 3$. (This comes from the Lefschetz Hyperplane Theorem; see for example [6, p. 159].) To see that (M, L) is spin^c bordant to a manifold of generalized positive scalar curvature, observe that $\Omega_6^{\text{spin}^c} \cong \mathbb{Z}^2$, and that the bordism class of any spin^c 6-manifold (M, L) is detected by $\langle c_1(L)^3, [M]\rangle$ and by $\langle c_1(L)p_1(M), [M]\rangle$. From the formulas in Proposition 1.4, we see that $7h - 14z$ is an allowable spin^c line bundle on M, and since $p_1 = -24\widehat{A}$ in degree 4 cohomology, spin^c bordism classes of 6-manifolds are determined by $\langle c_1(L)^3, [M]\rangle$ and by $\alpha^c(M, L)$. Now $\alpha^c(Q, h) = 0$ and $\langle h^3, [Q]\rangle = 2$ (since Q has degree 2 in $\mathbb{CP}^4$). On the other hand, for the given choice of L with $c_1(L) = 7h - 14z$, $c_1(L)^3 = 7^3h^3$, so $\langle c_1(L)^3, [M]\rangle = 2\cdot 7^3 = 686$ and

$$\alpha^c(M, L) = \frac{7\cdot(7^2-1)}{24} - 14 = 0.$$

That shows that (M, L) is spin^c bordant to a disjoint sum of 343 copies of (Q, h), which admits generalized positive scalar curvature by Theorem 1.2. (Alternatively, in the notation of Definition 2.1 below, $[(Q, h)] = T_{\mathsf{MSpin}^c}(\mathbb{CP}^1, \mathcal{O}, \mathcal{O}^2\oplus\mathcal{O}(1))$.) On the other hand, $L = E^{\otimes 7}$ with $c_1(E) = h - 2z$, and again by Proposition 1.4, we have $\alpha^c(M, E) = 0 - 2 = -2 \neq 0$, and we can apply Lemma 1.3 to get the conclusion. □

2. The Transfer Maps

There are some technical errors in [2, Theorem 5.1] and its proof. The statement of this theorem is correct, provided one slightly modifies the definition of the transfer map T_{MSpin^c}. This map should be defined using $U(3)$ instead of $SU(3)$, in order to have $\ker\alpha^c\otimes\mathbb{Z}_{(p)} = \operatorname{im} T_{\mathsf{MSpin}^c}\otimes\mathbb{Z}_{(p)}$ for every odd prime p. (This is not true for $p = 3$ using the definition given in [2, §5], since $\pi_1(PU(3)) \cong \mathbb{Z}/3$, and with the definition given in [2], the image of the transfer map in degree 6 is generated by $(\mathbb{CP}^1, \mathcal{O}(2))\times(\mathbb{CP}^2, \mathcal{O}(1))$, which has characteristic numbers $(c_1^3, p_1c_1) = (6, 6)$, whereas the quadric complex 3-fold (Q, h) from Section 1 also lies in the kernel of α^c and has characteristic numbers $(c_1^3, p_1c_1) = (2, 2)$.)

The correct definition of T_{MSpin^c} is as follows.

Definition 2.1. Let (B, L_B) be a closed spin^c manifold and $V \to B$ a complex vector bundle of rank 3 (a $U(3)$-bundle). Let $\pi\colon \mathbb{P}(V) \to B$ be the bundle of lines in V and $\xi = c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$. Put

$$T(B, L_B, V) := \Big(\mathbb{P}(V),\ \mathcal{O}_{\mathbb{P}(V)}(1)\otimes\pi^*(L_B\otimes\det V)\Big),$$

with the spin^c structure described below in Lemma 2.2. This defines a homomorphism

$$T^c_{\mathsf{MSpin}} := T\colon\ \Omega_n^{\text{spin}^c}(BU(3)) \to \Omega_{n+4}^{\text{spin}^c}.$$

Lemma 2.2. *Let (B, L_B) and V be as in Definition 2.1, $E = \mathbb{P}(V)$, T_v the vertical tangent bundle.*

(a) $TE \oplus \mathbb{C} \cong \pi^* TB \oplus \pi^* V \otimes \mathcal{O}(1)$, $T_v \oplus \mathbb{C} \cong \pi^* V \otimes \mathcal{O}(1)$, *and* $c_1(T_v) = 3\xi + \pi^* c_1(V)$. *The spin^c structure on $\pi^* TB$ with determinant $\pi^* L_B$, the complex structure on $\pi^* V \otimes \mathcal{O}(1)$, and the twist by the line bundle $\mathcal{O}(-1)$ define a spin^c structure on E with determinant $L_E := \mathcal{O}(1) \otimes \pi^*(L_B \otimes \det V)$. It depends only on the bordism class of (B, L_B, V), so T is well defined.*

(b) $\operatorname{im} T$ *is an ideal of* $\Omega_*^{\text{spin}^c}$.

(c) $\alpha^c(T(B, L_B, V)) = 0$. *More generally, for* $n \in \mathbb{Z}$,

$$\operatorname{ind}\left(D_{L_E} \otimes L_E^{\otimes n}\right) = \left\langle \widehat{A}(B)\, e^{(2n+1)c_1(L_B)/2}\, e^{n\, c_1(V)}\ \operatorname{ch}\left(\pi_! \mathcal{O}(n-1)\right), [B] \right\rangle, \tag{1}$$

where $\pi_! \mathcal{O}(j) = \operatorname{Sym}^j V^*$ *for* $j \geq 0$, $= 0$ *for* $j = -1, -2$, *and* $= \operatorname{Sym}^{-3-j} V \otimes \det V$ *for* $j \leq -3$.

(d) $T(B, L_B, V)$ *admits gpsc.*

Proof. (a) The splitting of $TE \oplus \mathbb{C}$ is standard; $w_2(E) \equiv w_2(B) + c_1(V) + 3\xi \equiv c_1(L_E) \pmod 2$, and twisting a spin^c structure by a line bundle N multiplies its determinant by $N^{\otimes 2}$, which gives the claim about L_E.

(b) We have $(N, L_N) \times T(B, L_B, V) = T(N \times B, L_N \boxtimes L_B, \operatorname{pr}_B^* V)$.

(c) Since $\widehat{A}(T_v) e^{c_1(T_v)/2} = \operatorname{Td}(T_v)$,

$$\widehat{A}(E)\, e^{(2n+1)c_1(L_E)/2} = \widehat{A}(B)\, e^{(2n+1)c_1(L_B)/2}\, e^{n c_1(V)} \cdot \operatorname{Td}(T_v)\, e^{(n-1)\xi},$$

and $\pi_*\left(\operatorname{Td}(T_v) e^{(n-1)\xi}\right) = \operatorname{ch}(\pi_! \mathcal{O}(n-1))$ by the Grothendieck-Riemann-Roch theorem for the projective bundle; the description of $\pi_! \mathcal{O}(j)$ is the cohomology of $\mathcal{O}(j)$ on $\mathbb{CP}^2$ together with relative Serre duality ($K_{E/B} = \mathcal{O}(-3) \otimes \pi^* \det V^*$). For $n = 0$ we get $\pi_! \mathcal{O}(-1) = 0$.

(d) Exactly as in the proof of [2, Cor. 5.2]: $U(3)$ acts isometrically on $(\mathbb{CP}^2, g_{FS})$ and preserves $\mathcal{O}(1)$ with its Fubini–Study connection, so E carries submersion metrics with fibers $\epsilon^2 g_{FS}$ and L_E carries the tensor product of the fiberwise Fubini-Study connection with any connection on $\pi^*(L_B \otimes \det V)$; then $\frac{1}{4} \operatorname{scal} + R_{L_E} \geq 4\epsilon^{-2} - C > 0$ for ϵ small enough. □

Definition 2.3 ([2, §5])**.** The transfer map $T_{\mathsf{MSpin}} \colon \Omega_n^{\text{Spin}^c}(BP\mathrm{Sp}(3)) \to \Omega_{n+8}^{\text{Spin}^c}$ is the transfer map of Stolz [8], defined using $\mathbb{HP}^2$ bundles, smashed with $\Sigma^{-2}\mathbb{CP}^\infty$ (via the equivalence $\mathsf{MSpin}^c \simeq \mathsf{MSpin} \wedge \Sigma^{-2}\mathbb{CP}^\infty$).

Theorem 2.4 ([2, Theorem 5.1])**.** *The transfer maps T^c_{MSpin} and T_{MSpin} have images lying in the kernel of $\alpha^c \colon \mathsf{MSpin}^c \to \mathsf{ku}$ and generate* $\ker \alpha^c \colon \Omega_\bullet^{\text{Spin}^c} \to \mathbb{Z}$ *additively.*

Proof (*Corrected proof of* [2, Theorem 5.1])*.* We prove this separately after inverting 2 and localized at 2.

Proof at odd primes. After tensoring with $\mathbb{Z}\left[\frac{1}{2}\right]$, the natural map (assigning the spin^c structure given by the anti-canonical bundle) $\iota \colon \mathsf{MU} \to \mathsf{MSpin}^c$ is surjective [10, pp. 348–349], and is a ring map of ring spectra. The composition $\operatorname{Td} = \alpha^c \circ \iota$ is the Todd genus. Thus, after tensoring with $\mathbb{Z}_{(p)}$ for p an odd prime, $\ker \alpha^c$ is generated by stably almost complex manifolds with vanishing Todd genus. Now $\Omega_\bullet^U \otimes \mathbb{Z}_{(p)} \cong \mathbb{Z}_{(p)}[z_2, z_4, \cdots]$ with the polynomial generators z_{2j} given by almost complex manifolds of (complex) dimensions $j = 1, 2, 3, \cdots$. We can take $z_2 = [\mathbb{CP}^1, \mathcal{O}(2)]$, which has Todd genus 1. Since the kernel of α^c is an ideal, it suffices to show that we can construct the other polynomial generators z_{2j}, $j > 1$, to be in the image of T^c_{MSpin}. Now an almost complex manifold M of complex dimension j can be chosen to be a polynomial generator of $\Omega_{2j}^U \otimes \mathbb{Z}_{(p)}$ for a prime p exactly when $v_p(s_j(M)) = v_p(m_j)$, where v_p is the p-adic valuation and $m_j = p$ if $j + 1$ is a power of p and $m_j = 1$ otherwise [10, Ch. VII]. The number m_j always divides $s_j(x)$ for every x. The construction of the manifolds representing z_{2j}, $j > 1$, after localizing at a fixed prime $p > 2$ follows in Definition 2.5 and Lemmas 2.6 and 2.7.

Proof at the prime 2. We apply the basic outline of the proof in [2]. We have an additive splitting (here everything is after localizing at 2)

$$\begin{aligned}\mathsf{MSpin}^c &\simeq \mathsf{MSpin} \wedge \Sigma^{-2}\mathbb{CP}^\infty \\ &\simeq \big(\mathsf{ko} \vee (\text{higher summands in the image of } T_{\mathsf{MSpin}})\big) \wedge \Sigma^{-2}\mathbb{CP}^\infty. \\ &\simeq \big(\mathsf{ko} \wedge \Sigma^{-2}\mathbb{CP}^\infty\big) \vee \operatorname{im} T_{\mathsf{MSpin}}\end{aligned}$$

Now $\operatorname{im} T_{\mathsf{MSpin}}$ lies in the kernel of α^c, as explained in [2]. So we just need to show that the kernel of α^c inside $\mathsf{ko} \wedge \Sigma^{-2}\mathbb{CP}^\infty$ lies in the image of T_{MSpin^c}. It was also observed in [2] that

$$q\colon \mathsf{ko} \wedge \Sigma^{-2}\mathbb{CP}^\infty \simeq \bigvee_{k=0}^{\infty} \Sigma^{4k}\mathsf{ku}, \tag{2}$$

and that the summands on the right are some of the suspensions $\Sigma^{4n(J)}\mathsf{ku}$ of ku, indexed by partitions J of size $n(J)$, that appear in the Anderson–Brown–Peterson additive splitting of MSpin^c localized at 2. Here α^c can be identified with projection on to the lowest ($k = 0$) summand. So we need to show that $\bigvee_{k=1}^{\infty} \Sigma^{4k}\mathsf{ku}$ lies in the image of T_{MSpin^c}.

There was an inaccuracy in our proof in [2], since the statement that the transfer map "acts the same way on all the summands," and thus that we can reduce to the study of the transfer map on the bottom summand of ku, is questionable, since the splitting in (2) is not a splitting of ku-module spectra, as was shown in [9, §7]. (The argument there was about the Anderson–Brown–Peterson splitting of MSpin but would also apply to MSpin^c.) However, this can be fixed using the argument of Hopkins and Hovey in [7, §§4–5].

Thus to make the argument precise, consider the Anderson–Brown–Peterson splitting $s\colon \mathsf{ku} \to \mathsf{MSpin}^c$ to $\alpha^c\colon \mathsf{MSpin}^c \to \mathsf{ku}$. Recall that α^c a ring spectrum map but that s is only a (non-multiplicative) map of spectra. Let e_j be the class in $\Omega_{4j}^{\mathrm{spin}^c}$ of the $\mathbb{CP}^2$-bundle $E_j \xrightarrow{\pi} \mathbb{HP}^{j-1}$ obtained as $E_j = \mathbb{P}(\gamma \oplus \mathbb{C})$, γ the tautological $\mathbb{H}$-bundle over $\mathbb{HP}^{j-1}$ viewed as a $\mathbb{C}^2$-bundle, with spin^c line bundle ξ coming from $\mathcal{O}(1)$ on $\mathbb{CP}^2$. (Note that $E_1 = \mathbb{CP}^2$.) The e_j's lie in the image of T^c_{MSpin}, and thus have generalized positive scalar curvature, and it's easy to check that $\langle \xi^{2j}, E_j\rangle = \pm 1$, since $\pi_!(\xi^2) = 1 \in H^0(\mathbb{HP}^{j-1})$ and $\xi^3 = -c_2(\gamma)\xi$.

Let $h\colon \pi_*(\mathsf{ku}) \to H_*(\mathsf{ku}; \mathbb{F}_2)$ denote the (mod 2) Hurewicz map. Since $\pi_*(\mathsf{ku}) = \mathbb{Z}[\beta]$, β the Bott element in $ku_2 = \pi_2(\mathsf{ku})$, and the integral Hurewicz map is multiplication by 2 from $\pi_2(\mathsf{ku}) \cong \mathbb{Z}$ to $H_2(\mathsf{ku}) \cong \mathbb{Z}$ [1, p. 342], $h(\beta) = 0$ and thus h vanishes on all elements of positive degree. Similarly, any map $\Sigma^{2j}\mathsf{ku} \to \Sigma^{2k}\mathsf{ku}$ is zero on $\mathbb{F}_2$-cohomology if $j > k$. Furthermore, $H^*(\mathsf{ku}; \mathbb{F}_2) \cong \mathcal{A} \otimes_{E(1)} \mathbb{F}_2$ is a cyclic $\mathcal{A}$-module with a unique generator in degree 0 (here $\mathcal{A}$ is the mod-2 Steenrod algebra and $E(1)$ is the exterior algebra on $Q_0 = \mathrm{Sq}^1$ and $Q_1 = \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1$), so a map $\Sigma^{2j}\mathsf{ku} \circlearrowleft$ induces the identity on $H^*(\Sigma^{2j}\mathsf{ku}; \mathbb{F}_2)$ if the induced self-map on $\pi_{2j}(\Sigma^{2j}\mathsf{ku}) \cong \mathbb{Z}$ is odd, and induces the 0-map on $H^*(\Sigma^{2j}\mathsf{ku}; \mathbb{F}_2)$ if the self-map on π_{2j} is even.

Now observe that under the map q of (2), $h(q_*(e_j)) \neq 0$, because of the fact that $\langle \xi^{2j}, E_j\rangle = \pm 1$, since evaluating the top power of the spin^c line bundle on the fundamental class is what detects the Hurewicz image of the bottom of the j-th summand $\Sigma^{4j}\mathsf{ku}$ in (2). From what we've said about h, this implies that if $\mu\colon \mathsf{MSpin}^c \wedge \mathsf{MSpin}^c \to \mathsf{MSpin}^c$ is the multiplication, $e_0 = 1$, and we define

$$f_j\colon \Sigma^{4j}\mathsf{ku} \xrightarrow{e_j \wedge s} \mathsf{MSpin}^c \wedge \mathsf{MSpin}^c \xrightarrow{\mu} \mathsf{MSpin}^c \xrightarrow{q} \bigvee_{k=0}^{\infty} \Sigma^{4k}\mathsf{ku}, \qquad j \geq 0,$$

where $e_j \in \pi_{4j}(\mathsf{MSpin}^c)$ is viewed as a [homotopy class of a] map $S^{4j} \to \mathsf{MSpin}^c$, then for $j = 0$ the composite is homotopic to qs. Set

$$\Phi = \bigvee_{j\geq 0} f_j \colon \bigvee_{j\geq 0} \Sigma^{4j}\mathsf{ku} \longrightarrow \mathsf{ko} \wedge \Sigma^{-2}\mathbb{CP}^\infty \simeq \bigvee_{k\geq 0} \Sigma^{4k}\mathsf{ku}. \tag{3}$$

(This is the bottom-cell multiplication construction of [7, §5, pp. 189–190], with the classes e_j supplying the bottoms.) Then Φ is an equivalence of spectra, since f_j has non-zero mod-2 Hurewicz image only onto the $k = j$ summand.

Since the e_j's lie in the image of the transfer map T_{MSpin^c}, this patches the proof in [2]. □

Definition 2.5. Let B be a closed stably almost complex manifold and N_1, N_2 complex line bundles on B, $n_i = c_1(N_i)$. Put $V = \mathcal{O} \oplus N_1 \oplus N_2$ and $E = \mathbb{P}(V)$. By Lemma 2.2(a),

$$TE \oplus \mathbb{C} \cong \pi^*TB \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)N_1 \oplus \mathcal{O}(1)N_2.$$

Let $Z = Z(B; N_1, N_2)$ denote E with the stable complex structure obtained by replacing the summand $\mathcal{O}(1)$ by its conjugate $\mathcal{O}(-1)$. Its underlying oriented manifold is $-E$, and $c_1(Z) = c_1(B) + \xi + n_1 + n_2$.

Lemma 2.6.

(a) *We have* $\iota(Z(B; N_1, N_2)) = -T(B, K_B^{-1}, \mathcal{O} \oplus N_1 \oplus N_2)$ *in* $\Omega_*^{\mathrm{spin}^c}$ *whenever* E *is simply connected; in particular* $\mathrm{Td}(Z) = 0$.

(b) *Let* $\dim_{\mathbb{C}} B = k - 2$, $k \geq 2$. *Then* $s_k(Z) = -\langle \Phi_k(n_1, n_2), [B]\rangle$ *where, with* $\pi_*\xi^{j+2} = (-1)^j h_j(n_1, n_2)$ (h_j *the complete symmetric polynomial*),

$$\begin{aligned} \Phi_k(n_1, n_2) &= \pi_*\big((\xi + n_1)^k + (\xi + n_2)^k + (-\xi)^k\big) = \sum_{i=0}^{k-2} \phi_i\, n_1^i n_2^{k-2-i}, \\ \phi_i &= 1 + (-1)^{k-i}\tbinom{k-1}{i} + (-1)^i\tbinom{k-1}{i+1}. \end{aligned}$$

(c) *For* $Z_i := Z(\mathbb{CP}^i \times \mathbb{CP}^{k-2-i}; \mathcal{O}(1,0), \mathcal{O}(0,1))$, $0 \leq i \leq k-2$, *one has* $s_k(Z_i) = -\phi_i$.

Proof. (a) $\iota(Z)$ is $-E$ with the spin^c structure of determinant $\mathcal{O}(1) \otimes \pi^*(K_B^{-1} \otimes N_1 \otimes N_2)$, which is L_E for $T(B, K_B^{-1}, V)$; on a simply connected manifold a spin^c structure is determined by its determinant and the orientation, so $\iota(Z)$ is the bordism inverse of $T(B, K_B^{-1}, V)$. Then $\mathrm{Td}(Z) = \alpha^c(\iota Z) = 0$ by Lemma 2.2(c). One can also check directly that

$$\begin{aligned} &\pi_*\big(\mathrm{Td}(\mathcal{O}(-1))\,\mathrm{Td}(\mathcal{O}(1)N_1)\,\mathrm{Td}(\mathcal{O}(1)N_2)\big) \\ &\qquad = G(n_1)G(n_2) - G(n_1)G(n_2 - n_1) - G(n_2)G(n_1 - n_2) = 0 \text{ for } G(t) = (1 - e^{-t})^{-1}. \end{aligned}$$

(b) The Chern roots of Z are those of π^*TB (which contribute nothing to s_k for dimensional reasons) together with $-\xi$, $\xi + n_1$, $\xi + n_2$; and $[Z] = -[E]$. The relation $\xi(\xi + n_1)(\xi + n_2) = 0$ with $\pi_*\xi^2 = 1$ gives $\pi_*\xi^{j+2} = (-1)^j h_j(n_1, n_2)$. Expanding,

$$\Phi_k = \sum_{m=2}^{k} \tbinom{k}{m}(-1)^m h_{m-2}(n_1, n_2)\big(n_1^{k-m} + n_2^{k-m}\big) + h_{k-2}(n_1, n_2),$$

and reading off the coefficient of $n_1^i n_2^{k-2-i}$ with $\sum_{m=j}^{k}(-1)^m\binom{k}{m} = (-1)^j\binom{k-1}{j-1}$ gives the stated ϕ_i. (c) is immediate from (b). For instance $\Phi_3 = 2(n_1 + n_2)$ and $\Phi_4 = 5(n_1^2 - n_1n_2 + n_2^2)$. □

Lemma 2.7. *Let* $k \geq 2$ *and let* p *be an odd prime. Then* $\min\limits_{0\leq i\leq k-2} v_p(\phi_i) = v_p(m_k)$.

Proof. It suffices to show $\min_i v_p(\phi_i) \le v_p(m_k)$, as the reverse inequality comes for free.

• *k even.* Then $\phi_i = 1 + (-1)^i\binom{k}{i+1}$ and $m_k = p$ iff $k+1 = p^j$. If $p \mid \phi_i$ for all i, then $\binom{k}{m} \equiv (-1)^m \pmod p$ for $1 \le m \le k-1$, hence $\binom{k+1}{m} \equiv 0$ for $1 \le m \le k$, i.e. $(1+x)^{k+1} \equiv 1+x^{k+1} \pmod p$, which forces $k+1 = p^j$. So if $k+1$ is not a power of p some ϕ_i is a p-adic unit. If $k+1 = p$, then $\phi_0 = 1 + \binom{p-1}{1} = p$. If $k+1 = p^j$ with $j \ge 2$ and $p^2 \mid \phi_i$ for all i, the same computation gives $\binom{p^j}{m} \equiv 0 \pmod{p^2}$ for $2 \le m \le p^j - 2$; but $v_p\left(\binom{p^j}{p^{j-1}}\right) = 1$ by Kummer's theorem, a contradiction since $2 \le p^{j-1} \le p^j - 2$. Hence some ϕ_i has $v_p(\phi_i) = 1$.

• *k odd.* Then $k \ge 3$, $m_k = 1$, and $\phi_i = 1 - (-1)^i\binom{k-1}{i} + (-1)^i\binom{k-1}{i+1}$. If $p \mid \phi_i$ for all i, the coefficients of $(1+x)^{k-1}(1-x)$ in degrees $1, \dots, k-1$ are $\equiv (-1)^m \pmod p$, so

$$(1+x)^{k-1}(1-x) \equiv \sum_{m=0}^{k} (-1)^m x^m = (1-x^{k+1})/(1+x),$$

i.e. $(1+x)^k(1-x) \equiv 1 - x^{k+1} \pmod p$. Comparing the coefficients of x and x^2 gives $k \equiv 1$ and $k(k-3)/2 \equiv 0$, i.e. $-1 \equiv 0 \pmod p$, a contradiction. □

3. Implications for Other Work

As far as the rest of [2] is concerned, very little needs to be changed. Theorem 1.1 was used in the rest of [2] only in the proof of Theorem 3.20 (the even-type case), applied to M_Σ with the spin^c line bundle L_Σ, which is trivial on M and equals $\tilde{p}^*L$ on the tubular neighborhood $D(L)$ of βM [2, Theorem 3.12]; in particular $L_\Sigma|_{\beta M} = L$, the line bundle of the circle bundle $p\colon \partial M \to \beta M$. If each component of ∂M is simply connected, then Theorems B, 3.19, 3.20 of [2] work as stated, since βM will also be simply connected.

In [3, Theorem 1.2], it was asserted that vanishing of $\alpha^c(M, L)$ is necessary and sufficient for a non-spin simply connected spin^c manifold to admit generalized positive scalar curvature. Proposition 1.5 shows that this assertion is incorrect. However, it's true in the case where the classifying map to MSpin^c is a 2-equivalence. This is all that's needed for the example in [3, §6].

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