A Gentle Introduction to HyperKähler Manifolds

Geometry/Physics RIT

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Kähler Manifolds

First, some basic definitions:

- An *almost complex structure* is a linear map $I \in End(TX)$ such that $I^2 = -1$ on the tangent spaces, varying smoothly over the manifold.
- An almost complex structure is *integrable* or a *complex structure* if there is a covering of X by open sets U_α with holomorphic diffeomorphisms $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}^n$. If I is integrable, then (X,I) is a complex manifold.
- A *Hermitian metric* on (X,I) is a Riemannian metric satisfying

$$g(v, w) = g(Iv, I, w).$$

- For g a Hermitian metric on (X,I), the *fundamental form* is the (1,1)-form ω such that

$$\omega(v, w) = g(Iv, w).$$

Kähler Manifolds

Definition

If g is a Hermitian metric on (X,I), we say g is a Kähler metric if

$$\nabla I = 0$$

where ∇ is the Levi-Civita connection on TX induced by g. If g is a Kähler metric, then (X,I,g) is a Kähler manifold.

Kähler Form

For (X,I,g) a Kähler manifold, denote the fundamental 2-form ω, called the Kähler form.

- ω is a (1,1)-form, written locally as

$$\omega = i \sum_{i,j=1}^{n} g_{ij}(z) dz_i \wedge d\bar{z}_j$$

- ω is closed
- ω is positive definite: the matrix $(g_{ij}(z))$ is a positive definite hermitian matrix for every z. In particular, ω is non degenerate.

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Observation:

For (X, I, g) Kähler, ω is a closed nondegenerate 2-form on X, making X a symplectic manifold

Examples of Kähler Manifolds

Kähler manifolds are *abundant*:

- $\mathbb{C}^n$ is Kähler, with

$$\omega = i \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$$

- $\mathbb{CP}^n$ is Kähler, with

$$\omega = i \frac{(1 + |z|^2) \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i - \sum_{i,j=1}^{n} \bar{z}_i z_j dz_i \wedge d\bar{z}_j}{(1 + |z|^2)^2}$$

- Every Riemann surface is Kähler, with a volume form $f(x, y) dx \wedge dy$ which can be written as $g(z) dz \wedge d\bar{z}$, which is a Kähler form.

HyperKähler Manifolds

HyperKähler manifolds extend the notion of Kähler manifolds to manifolds based on the algebra of quaternions.

Definition

A *hyperKähler manifold* is a tuple (X, g, I, J, K), where (X, g) is a Riemannian manifold equipped with complex structures I, J, and K with

$$I^2 = J^2 = K^2 = IJK = -1$$

such that X is Kähler with respect to each structure.

Then, denote the associated Kähler forms by ω_I, ω_J, ω_K.

Kähler Vs. HyperKähler

This is certainly a stronger condition than Kähler, which only requires a single automorphism I satisfying $I^2 = -1$.

Consequentially, hyperKähler manifolds and metrics are far less abundant:

- A Kähler metric on M may be modified to another by addition of a Hermitian form $\partial\bar{\partial}f$ for a small $C^\infty f$, giving an infinite dimensional space of Kähler metrics. In contrast, if a hyperKähler metric exists, then there is only a finite dimensional space.
- Kähler manifolds themselves are plentiful (as noted above), in particular, complex submanifolds of Kähler manifolds are Kähler. HyperKähler manifolds are relatively scarce, and constructing them can be challenging

Holomorphic Symplectic Manifolds

Definition

If (X,I) is a complex manifold, a (2,0)-form Ω on X is a *holomorphic symplectic form* if $d\Omega = 0$ and Ω is nondegenerate (induces an isomorphism $T^{(1,0)}X \to \left(T^{(1,0)}X\right)^*$). For such a form, (X, I, Ω) is a holomorphic symplectic form.

Such manifolds have real dimension $4n$ for some n. Further, holomorphic symplectic manifolds locally admit Darboux coordinates $(p_1, ..., p_n, q_1, ..., q_n)$ such that

$$\Omega = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

Holomorphic Symplectic Form Determines Complex Structure

Theorem

Suppose X is a manifold with real dimension 4n, and holomorphic 2-form Ω such that $d\Omega = 0$ and

$$T_{\mathbb{C}}X = ker\Omega \oplus ker\bar{\Omega}.$$

Then there is a unique complex structure I on X for which Ω is a holomorphic symplectic form.

The Berger Classification

Hol(g)	dim(M)	Type of manifold	Comments
SO(n)	n	Orientable manifold	—
U(n)	$2n$	Kähler manifold	Kähler
SU(n)	$2n$	Calabi–Yau manifold	Ricci-flat, Kähler
Sp(n) · Sp(1)	$4n$	Quaternion-Kähler manifold	Einstein
Sp(n)	$4n$	Hyperkähler manifold	Ricci-flat, Kähler
G_2	7	G_2 manifold	Ricci-flat
Spin(7)	8	Spin(7) manifold	Ricci-flat

Figure: Berger Classification (via Wikipedia)

Holonomy of HyperKähler Manifolds

On a hyperKähler manifold, parallel translation preserves I, J, and K (since they are covariantly constant) so $Hol(g) \subseteq O(4n)$ and $Hol(g) \subseteq GL(n, \mathbb{H})$ (quaternionic invertible matrices). The maximal intersection is Sp(n), the group of $n \times n$ quaternionic unitary matrices.

HyperKähler Manifolds are Holomorphic Symplectic

Sp(n) is also an intersection of U(2n) and $sp(2n, \mathbb{C})$ (the linear transformations of $\mathbb{C}^{2n}$ which preserves a nondegenerate skew form). then, a hyperKähler manifold is a complex manifold with holomorphic symplectic forms given by the Kähler 2-forms

$$\Omega_I = \omega_J + i\omega_K$$

(and cyclic permutations).

HyperKähler Forms Determine Complex Structure

Theorem

If X is hyperKähler, then

$$I = \omega_K^{-1}\omega_J$$

and analogous for J and K.

Thinking of ω_J as a map $TX \to T^*X$ defined by $v \mapsto \omega_J(v, \cdot)$ and ω_K^{-1} as a map $T^*X \to TX$ with $\omega_K(v, \cdot) \mapsto v$, then $\omega_K^{-1}\omega_J : TX \to TX$ is the complex structure I.

Proof: That is, show $\omega_K(Iv, \cdot) = \omega_J(v, \cdot)$. But this is immediate, as

$$\omega_K(Iv, \cdot) = g(KIv, \cdot) = g(Jv, \cdot) = \omega_J(v, \cdot)$$

Forms on a Manifold Determine the Metric

HyperKähler Forms Determine the Metric

If X is hyperKähler, then

$$g = -\omega_I \omega_K^{-1} \omega_J$$

(and cyclic permutations)

Further, given a holomorphic symplectic forms can can be used to find a hyperKähler metric on a manifold not assumed to be hyperKähler:

Symplectic Forms Give HyperKähler Metric

Suppose X is a manifold with symplectic forms $\omega_1, \omega_2, \omega_3$ such that

$$-\omega_1 \omega_3^{-1} \omega_2 = -\omega_2 \omega_1^{-1} \omega_3 = -\omega_3 \omega_2^{-1} \omega_1 = g$$

and that g is a positive definite symmetric bilinear form. Then g is a hyperKähler metric on X with Kähler forms ω_i.

A HyperKähler Form is Kähler for S^2-Many Complex Structures

If a metric is Kähler for three complex structures, then it Kähler for S^2-many complex structures.

Theorem

Suppose (X, g, I_1, I_2, I_3) is a hyperKähler manifold. Then for any $\vec{s} = (s_1, s_2, s_3) \in S^2 \subset \mathbb{R}^3$, define

$$I_{\vec{s}} = \sum_{i=1}^{3} s_i I_i \qquad \omega_{\vec{s}} = \sum_{i=1}^{3} s_i \omega_i .$$

Then $(X, g, I_{\vec{s}})$ is a Kähler manifold with form $\omega_{\vec{s}}$.

Thus, specifying three complex structures is equivalent to specifying a whole family $I_{\vec{s}}$, $\vec{S} \in S^2$.

Existence of HyperKähler Metrics

For a compact Kähler manifold with holomorphically trivial canonical bundle, by Calabi-Yau there is a Kähler metric with vanishing Ricci tensor. Then, by another theorem (Bochner), any holomorphic form on a compact Kähler manifold with zero Ricci tensor is covariantly constant. It follows from these results that any compact Kähler manifold with a holomorphic symplectic form, there is a hyperKähler metric on this manifold.

This is a very useful criterion for showing the existence of hyperKähler metrics on certain complex manifolds, notably the K3 surface. However, it only shows existence, and does not explicitly give the structure.

This is done, by obtaining some differential equation, and solving for the metric.
There are two main methods for constructing hyperKähler metrics:

- Twistor Theory: This essentially encodes the data for the metric in terms of holomorphic geometry. In a sense, this reduces the differential equations to the Cauchy-Riemann equation, but explicitly writing the metric is still challenging in general. Deriving global properties of metrics obtained by this method is very challenging, but the following method makes these properties more accessible.
- HyperKähler Quotients:

Example—Flat Space

$\mathbb{H}$ is a four dimensional real vector space under the natural identification $x_0 + x_1 i + x_2 j + x_3 k \mapsto (x_0, x_1, x_2, x_3)$, -and so we may identify $T_p\mathbb{H} \simeq \mathbb{H}$ to get almost complex structures (I,J,K), and the norm $\|q\|^2 = q\bar{q}$ induces a metric g on $\mathbb{H}$.

Then, left multiplication by i, j, and k give complex structures I, J, K on $\mathbb{H}$. These are covariantly constant, so g is Kähler for each, and thus $\mathbb{H}$ is hyperKähler.

References:

This presentation was based on

- *Moduli of Higgs Bundles* by Neitzke
- *HyperKähler Manifolds* by Hitchin