MANIFOLDS OF POSITIVE SCALAR CURVATURE

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ABSTRACT. We survey the status of the problem of determining which differentiable manifolds (without boundary) have Riemannian metrics of positive scalar curvature. Of course, if the manifold is non-compact, one requires the metric to be complete.

CONTENTS

1. THE YAMABE PROBLEM AND THE TRICHOTOMY THEOREM

The study of manifolds of positive scalar curvature can be traced back to work related to the **Yamabe problem**, which in turn is one way of generalizing to higher dimensions the classical **uniformization theorem** for compact surfaces. For completeness, we give a formulation of the latter, but stated as a result in Riemannian geometry rather than as a result about Riemann surfaces. Hereafter, we will use the adjective "closed" for manifolds to mean "compact and without boundary," and to mean "connected" as well unless we state otherwise. The term "manifold" in this paper will always mean a C^∞ manifold.

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Uniformization theorem. *Let (M, g) be a closed Riemannian 2-manifold. Then there is a function $u \in C^\infty(M)$ such that $(M, e^u g)$ has constant Gaussian curvature. In other words, there is a metric in the same conformal class with constant curvature.*

There is also a uniqueness statement about u, but it's not relevant for our purposes. However, we note that by the Gauss-Bonnet Theorem, applied either to M or (if M is non-orientable) to its oriented double cover, the sign of the Gaussian curvature of the conformal metric $e^u g$ is an invariant of the diffeomorphism class of M.

Instead of starting with a fixed metric g on M, we can begin instead with a fixed differentiable surface and ask what nice Riemannian metrics can be put on it. Reformulating the above then gives:

Corollary. *Each diffeomorphism class of closed surfaces belongs to one and only one of the following classes:*
- *(a) spherical surfaces–those with a Riemannian metric of constant curvature 1;*
- *(b) flat surfaces–those with a Riemannian metric of vanishing curvature;*
- *(c) hyperbolic surfaces–those with a Riemannian metric of constant curvature -1.*

The classification of closed surfaces gives us a complete list of the surfaces in each class. Class (a) consists of S^2 and $\mathbb{RP}^2$, class (b) consists of T^2 and the Klein bottle, and class (c) contains a countable infinity of diffeomorphism types. Note also that classes (b) and (c) may be grouped together into the class of **aspherical surfaces**, those for which the universal cover is contractible. The aspherical surfaces are exactly those which do **not** admit a metric with positive Gaussian curvature.

It would of course be nice to have a generalization of the uniformization theorem to higher dimensions. But in dimensions > 2, there are several different competing notions of curvature (e.g., the Riemann curvature operator, sectional curvature, Ricci curvature, and scalar curvature). A little thought shows that there's no hope to replace the Gaussian curvature in the statement of the uniformization theorem by the sectional curvature, or even by the Ricci curvature, in dimensions > 2. However, Yamabe conjectured, and thought that he had proved, that the statement of the uniformization theorem still holds for closed manifolds of all dimensions if one replaces the Gaussian curvature by the scalar curvature. The **scalar curvature** is the weakest (local) curvature of a Riemannian manifold; it is the contraction of the Ricci curvature tensor. In dimension 2, it is twice the Gaussian curvature; in general, it measures the leading term in the difference between the volume of a small geodesic ball in the manifold and the volume of a ball of the same radius in a Euclidean space of the same dimension. While

Yamabe's original proof turned out to be wrong, attempts to rectify the proof came to be known as the **Yamabe problem**, and eventually led to a correct (but quite difficult) proof of the conjecture, due largely to work of T. Aubin and R. Schoen [Schn]. For surveys of much of the literature, see [Au, Chapter 6] and [Kaz].

The work on the Yamabe problem suggested asking more generally which functions $\kappa \in C^\infty(M)$ can be the scalar curvature of *some* metric on a given manifold M, and also showed that the sign of κ can be of crucial importance. A remarkable theorem of Kazdan and Warner extends the trichotomy of the corollary above to arbitrary dimensions.

Trichotomy theorem [KW1], [KW2]. *Let M^n be a closed differentiable manifold of dimension n. Then M belongs to exactly one of the following three classes:*

(a) *manifolds admitting at least one Riemannian metric for which the scalar curvature κ is non-negative but not identically zero (in this case, there exists a metric with $\kappa > 0$);*
(b) *manifolds not admitting a Riemannian metric with $\kappa > 0$, but admitting a metric with $\kappa \equiv 0$ (in this case, such a metric is Ricci-flat);*
(c) *manifolds not admitting a Riemannian metric with $\kappa \geq 0$, but admitting a metric with $\kappa < 0$.*

In fact, if $n \geq 3$, then the set of functions in $C^\infty(M)$ which can be the scalar curvature of some metric is exactly all of $C^\infty(M)$ in case (a), all κ which are either identically 0 or else negative somewhere in case (b), and all κ which are negative somewhere in case (c).

This theorem of course suggests an important problem: that of classifying the manifolds in these three classes, or more precisely, of giving necessary and sufficient conditions for a given manifold M to lie in the various classes. At the moment, examples are known in each dimension of manifolds in each of the classes, but there are no good conjectures about how to describe class (b) (or in other words, how to describe those Ricci-flat manifolds that cannot be given metrics of positive scalar curvature). Aside from Bieberbach manifolds, which admit flat metrics and are known not to admit positive scalar curvature, the main examples known in this class are Calabi-Yau manifolds, i.e., complex Kähler manifolds with $c_1 = 0$, for example, a smooth complex hypersurface of degree $n+1$ in $\mathbb{CP}^n$. Such a hypersurface has vanishing w_2, and when n is odd, it has non-vanishing $\hat{A}$-genus [Br], and so, as will be pointed out below, cannot belong to class (a). Since it has a Calabi-Yau metric, it belongs to class (b). For manifolds known not to be in class (a), V. Mathai [M] has shown how one can sometimes also prove that the manifold is not is class (b). For instance, if $w_2(M^n) = 0$ and $|\hat{A}(M)| > 2^{n/2}$, then M must be in class (c). Most of the rest of this survey will be devoted to the problem of describing the manifolds in class (a). For simplicity, we call these simply manifolds of positive scalar curvature.

2. Obstructions for Closed Manifolds to Admit Metrics of Positive Scalar Curvature (A) Dirac Operator Methods

The single most important tool for showing that certain manifolds do **not** have metrics of positive scalar curvature is based on the Dirac operator. A good basic reference about this operator and its properties is [LM]. There are by now many variants of the "Dirac operator technique," several of which will be used in §§5, 6, and 7 below, but the key observation all of these depend on is a Weitzenböck-type formula discovered by Lichnerowicz [Li]. In its simplest form, the identity says that that if D is the Dirac operator on a Riemannian spin manifold with scalar curvature κ, then $D^2 = \nabla^*\nabla + \frac{\kappa}{4}$, where ∇ is the covariant derivative operator on sections of the spinor bundle, and ∇^* is its adjoint with respect to the Hilbert space structure induced by the Riemannian metric. Thus when κ is strictly positive, so is D^2, and the spectrum of D is bounded away from zero.

This fact has a number of immediate consequences. The simplest, noted by Lichnerowicz [Li], is that if M^{2k} is a closed spin manifold of even dimension, then the index of D, viewed as an operator from sections of the positive half-spinor bundle S^+ to sections of the negative half-spinor bundle S^-, must vanish. (Since D is self-adjoint as an operator on sections of $S^+ \oplus S^-$, $(D|_{S^+})^*$ is the same as $D|_{S^-}$, and neither $D|_{S^+}$ nor its adjoint can have non-zero kernel.) On the other hand, this index can be computed by the Atiyah-Singer index theorem to be $\hat{A}(M)$, the $\hat{A}$-genus. So if the dimension of M is divisible by 4 and $\hat{A}(M) \neq 0$, the assumption that M has a metric of positive scalar curvature leads to a contradiction.

This observation was generalized by Hitchin [Hi], who noted that the canonical KO-orientation of spin manifolds can be interpreted in terms of a more general index theory for the Dirac operator. In fact, there is a map of spectra [ABP1], which by slight abuse of notation we will call

$$D : \mathrm{MSpin} \to bo,$$

such that the induced map $D_* = \alpha : \Omega_n^{\mathrm{spin}} \to bo_n$ takes the bordism class of a closed spin manifold M^n to the generalized index (with values in bo_n) of the Dirac operator D on M. Here it does not matter much whether one uses connective or periodic K-theory, since there is a natural map of spectra $p : bo \to KO$ which is an isomorphism on π_n, $n \geq 0$. As before, the index vanishes if the spectrum of D is bounded away from zero, so we obtain:

Theorem 2.1 (Lichnerowicz-Hitchin). *If M^n is a closed spin manifold with a metric of positive scalar curvature then $\alpha(M) = 0$ (if n is divisible by 4, this just means $\hat{A}(M) = 0$).*

While this is as much as one can get out of the Dirac operator in the case of closed simply-connected manifolds, there are various ways of adapting this argument to non-simply-connected manifolds or to complete manifolds. We deal here with the former; the latter require other kinds of index theory and will be treated in §7 below.

When M is not simply connected, any metric on M can be lifted in a locally isometric way to any covering of M, so any obstruction to positive scalar curvature on a covering gives an obstruction to positive scalar curvature on M itself. More profoundly, however, the presence of a fundamental group implies that M admits flat (or possibly 'almost flat' [CGM]) bundles E coming from representations, either finite- or infinite-dimensional, of the fundamental group π. Putting a connection on E makes it possible to define the 'Dirac operator D_E with coefficients in E,' and if the connection on E is actually flat, one again obtains the identity $(D_E)^2 = \nabla_E^* \nabla_E + \frac{\kappa}{4}$. If E is not flat, there is a correction term coming from the curvature of E, but if this curvature is sufficiently small compared with κ, the conclusion that the spectrum of D_E is bounded away from 0 still holds. Gromov and Lawson [GL1] considered the case where E is an ordinary complex vector bundle which is not flat, but such that one can make its curvature sufficiently small (by replacing the original M by a suitable large cover). Then the Atiyah-Singer theorem gives as before that $\langle \hat{\mathbb{A}}(M) \cdot \mathrm{ch}(E), [M] \rangle = 0$, where $\hat{\mathbb{A}}$ is the total $\hat{A}$-class, if M is a spin manifold with positive scalar curvature. Consequences include the impossibility of positive scalar curvature on certain aspherical or 'enlargeable' manifolds.

Another version of the same idea was introduced by the first author in [R1], [R2], [R3], and [R4]. In this case, E is taken to be literally flat, but in general is not an ordinary vector bundle (as flat ordinary vector bundles tend to be topologically trivial or at least to have vanishing characteristic classes). Instead, one works with bundles whose fibres are finitely generated projective modules over a C*-algebra, in this case $C^*_{\mathbb{R}}(\pi)$ or $C^*_{\mathbb{R},r}(\pi)$, where π is the fundamental group of M or a suitable group that $\pi_1(M)$ maps to. These are C*-completions of the real group ring $\mathbb{R}\pi$. A suitable index theory for elliptic operators with coefficients in such bundles was introduced by Miščenko and Fomenko, along with an appropriate generalization of the Atiyah-Singer Theorem. In the finest version of the theory, torsion invariants are taken into account and we recapture all possible generalizations of Hitchin's theorem (2.1) coming from flat bundles. For the most elegant formulation of the theory, we make use of a natural 'assembly map' A: $KO_*(B\pi) \to KO_*(C^*_{\mathbb{R}}(\pi))$, essentially introduced by Kasparov [K] following earlier ideas of Miščenko. For a homotopy-theoretic description of this map, following ideas of Loday [Lo], see [R4, Theorem 2.2]. The result (in the case where M admits a spin structure) is the following:

Theorem 2.2 ([R3]). *If π is a group, M^n is a closed spin manifold and $f: M \to B\pi$ is a continuous map, let $\alpha(M) = A(p(D_*([M, f])))$, where $[M, f]$ is the bordism class of $M \xrightarrow{f} B\pi$ in $\Omega_n^{\text{spin}}(B\pi)$. Then if M admits a metric of positive scalar curvature, $\alpha(M) = 0$ in $KO_n(C^*_{\mathbb{R}}(\pi))$.*

The element $\alpha(M)$, in the case where we take $f : M \to B\pi_1(M)$ to be the classifying map for the universal cover of M, represents the total obstruction to positive scalar curvature coming from the Miščenko-Fomenko index theory of the Dirac operator. When $\pi_1(M)$ is finite, its real group C*-algebra is a finite-dimensional semi-simple algebra over $\mathbb{R}$, and $\alpha(M)$ can also be interpreted as the total obstruction to positive scalar curvature coming from the analogue of Theorem 2.1 for Dirac operators on flat real vector bundles. Of course, Theorem 2.2 only has value when the assembly map A can be computed. The *strong Novikov conjecture* for π, which is plausible for π torsion-free, is that A is an isomorphism, or at least an injection. This is known to be true for many torsion-free groups of geometrical interest—see [K] and [R3]. The strong Novikov conjecture implies that the obstruction class $\alpha(M)$ can be replaced by $D_*([M, f]) \in KO_n(B\pi)$. Even weaker statements would imply that no aspherical closed manifold admits a metric of positive scalar curvature(see [R1]). When π is finite, A is far from being an isomorphism but is still often non-trivial (assuming π is of even order); it is computed explicitly in [R4].

A few other authors have given various adaptations of the Dirac operator technique, using other kinds of index theory. The results tend to be similar in spirit to those of [GL1] but valid under somewhat different, often much weaker, hypotheses. Without going into details, we cite as references the papers [GL3], [Mo], [CM], [CGM].

(B) Minimal Surface Methods

The relevance of minimal surface theory to the positive scalar curvature problem comes from a clever observation of Schoen and Yau ([SY1] and [SY2]). Suppose M^n is a closed manifold of positive scalar curvature κ and H^{n-1} is an oriented hypersurface which minimizes area (i.e., $(n-1)$-dimensional volume) in its homology class. Let $\tilde{\kappa}$ be the scalar curvature of H. Then the Gauss curvature equation (which relates the curvatures of M and H), together with the fact that the second variation of area must be non-negative, implies the inequality

([SY2, (1.6)]) $$\int_H \frac{\tilde{\kappa}\phi^2}{2} > -\int_H |\nabla\phi|^2$$

for all functions $\phi \in C^\infty(H)$ not vanishing identically. This is not enough to deduce that $\tilde{\kappa}$ must be positive, but it does imply (if $n > 3$) that one can make a conformal change in the metric on H (multiplying the metric by a power of the first eigenfunction of the modified Laplacian $\Delta - \frac{(n-3)\tilde{\kappa}}{4(n-2)}$) to give H positive scalar curvature. If $n = 3$, things are even easier: taking $\phi \equiv 1$ in the inequality above gives that the integral of the Gaussian curvature of H is positive, so that H is a sphere by Gauss-Bonnet. Thus if one knows for some reason that H cannot have positive scalar curvature, we obtain an obstruction to positive scalar curvature on M. Various consequences (not the best possible with these methods) are obtained in [SY1], [SY2], and in [GL3, §§8–12].

Minimal surface methods have a number of advantages and disadvantages over Dirac operator methods for constructing obstructions to positive scalar curvature. One clear advantage is that no spin structure is needed, so that w_2 never appears in the argument. Thus Gromov and Lawson [GL3, p. 186] were able to show that $\mathbb{CP}^2 \# T^4$ does not admit a metric of positive scalar curvature, in spite of the fact that none of its covers admit spin structures. Whether one could get the same result using suitable Dirac operators **with coefficients** on the universal cover (these **do** exist, in fact with coefficients in line bundles, since $\mathbb{CP}^2$ is a spin^c manifold) is still an open question.

On the negative side, to make use of the basic minimal surface technique, one needs to be able to find an area-minimizing oriented hypersurface in a given homology class in M. While any homology class is represented by an area-minimizing **integral current** H, such a current can have singularities (of codimension ≥ 7 in H). This is why the original Schoen-Yau paper was limited to the case $n \leq 7$; in this case H is automatically regular. Schoen and Yau have since announced [Yau] that they have a way of getting around this technical difficulty so as to extend the minimal surface technique to high dimensions. Intuitively, this sounds as if it shoudn't be so hard since the codimension of the singularities is big (cf. Theorem 3.1 below), but the details are presumably messy, and we haven't seen them.

3. The Surgery Theorem and Its Consequences

After discussing the obstructions to the existence of metrics of positive scalar curvature in the last section we now turn to the construction of such metrics. Of course, there are plenty of examples of Riemannian manifolds with positive scalar curvature: e.g. spheres, projective spaces (real, complex or quaternionic), or, more generally, quotients of compact semisimple Lie groups. Moreover, there are certain bundle constructions producing manifolds with metrics of positive scalar curvature (cf. §4). In fact, Lawson and Yau [LY] showed that any closed manifold M that admits an effective action of $SU(2)$ or $SO(3)$ can be given positive scalar curvature, and this was generalized by Lewkowicz [Le], who showed it is enough for M to **locally** admit actions of non-abelian Lie groups (of positive dimension), assuming these actions are "compatible" on the overlaps. Still, the manifolds constructed this way give a 'small' subset of the set of diffeomorphism classes of manifolds. The subset of diffeomorphism classes known to carry metrics of positive scalar curvature grew enormously when the following surgery theorem was proved independently by Gromov-Lawson and Schoen-Yau.

Let N be a manifold of dimension n and assume that there is an embedding of $S^k \times D^{n-k}$ in N. Let M be the manifold obtained by glueing the complement of $S^k \times \dot{D}^{n-k} \subset N$ ($\dot{D}^{n-k}$ is the open $(n-k)$-disk) and $D^{k+1} \times S^{n-k-1}$ along their common boundary $S^k \times S^{n-k-1}$. We say that M is obtained from N by *k-surgery* (or *surgery of codimension* $n-k$).

Surgery Theorem 3.1 (Gromov-Lawson [GL2] and Schoen-Yau [SY2]). *Let N be a closed manifold with positive scalar curvature metric, not necessarily connected, and let M be obtained from N by surgery of codimension ≥ 3. Then M has a positive scalar curvature metric.*

Note that the connected sum $M_1 \# M_2$ of two n-dimensional manifolds is obtained from the disjoint union of M_1 and M_2 by a 0-surgery. Hence a special case of (3.1) is:

Corollary 3.2. *Let M_1 and M_2 be closed manifolds of dimension $n \geq 3$ with a metric of positive scalar curvature. Then the connected sum $M_1 \# M_2$ admits a metric of positive scalar curvature.*

To apply the surgery theorem it is important to know when a closed manifold M can be obtained from another manifold N by a sequence of surgeries of codimension ≥ 3. It follows from Morse theory that M can be obtained by surgeries from N (with no condition on the codimension) if and only if M is bordant to N; i.e. if there is a compact manifold W whose boundary is the disjoint union of M and N [Mi2, Thm. 3.13]. It turns out that if we want to control the codimension of the surgeries involved we have to work with more sophisticated bordisms: Let $B \xrightarrow{p} BO$ be a fibration. A *B-manifold* is a smooth manifold M, embedded in some Euclidean space, together with a lift $\hat{\nu}: M \to B$ of the map $\nu: M \to BO$ classifying the normal bundle of M. We denote by $\Omega_n(B)$ the bordism group of n-dimensional manifolds with B-structure (cf. [Sw, §12] or [S, Chapter II]) and by $Pos_n(B)$ the subgroup of $\Omega_n(B)$ represented by B-manifolds $\hat{\nu}: M \to B$ such that M admits a metric of positive scalar curvature.

Bordism theorem 3.3. *Let $\hat{\nu}: M \to B$ be a B-manifold of dimension $n \geq 5$ such that $\hat{\nu}$ is a 2-equivalence (i.e. $\hat{\nu}$ induces an isomorphism on on π_0 and π_1 and a surjection on π_2). If $(M, \hat{\nu})$ represents a class in $Pos_n(B)$ then M admits a metric of positive scalar curvature.*

We want to point out that for a manifold M the map $\nu: M \longrightarrow BO$ can always be factored in the form $M \xrightarrow{\hat{\nu}} B(M) \xrightarrow{p} BO$ such that

(1) $\hat{\nu}$ is a 2-equivalence and
(2) p is a fibration with fibre F such that $\pi_n(F) = 0$ for $n \geq 2$.

In fact, the fibration p is uniquely determined up to homotopy equivalence by these properties. It is called the *normal 1-type* of M in [Kr].

Examples.

3.4. Assume that M is an oriented manifold with fundamental group π and $w_2(M) = 0$. Then ν can be factored in the form $M \xrightarrow{\hat{\nu}} B\pi \times \mathrm{BSpin} \xrightarrow{p} BO$, where $\hat{\nu} = f \times s$, f is the classifying map of the universal covering of M, s is a spin structure on M (considered as lift of ν through the projection $q: \mathrm{BSpin} \longrightarrow BO$) and p is the projection on the second factor composed with q.

3.5. Assume that M is an oriented manifold with fundamental group π and $w_2(\widetilde{M}) \neq 0$, where $\widetilde{M}$ is the universal covering of M. Then ν can be factored in the form $M \xrightarrow{\hat{\nu}} B\pi \times BSO \xrightarrow{p} BO$, where $\hat{\nu} = f \times s$, f is the classifying map of the universal cover of M, s is the orientation of M (considered as lift of ν through the projection $q: BSO \longrightarrow BO$) and p is the projection on the second factor composed with q.

3.6. Generalizing the example (3.4), assume that M is an oriented manifold with fundamental group π and $w_2(\widetilde{M}) = 0$. It follows that $w_2(M) = f^*(\beta)$ for some cohomology class $\beta \in H^2(B\pi; \mathbb{Z}/2)$. Following [KwSc, §2] let $Y(\pi, \beta)$ be the pullback of $\beta: B\pi \longrightarrow K(\mathbb{Z}/2, 2)$ and the second Stiefel-Whitney class $w_2: BSO \longrightarrow K(\mathbb{Z}/2, 2)$. Then ν can be factored in the form $M \xrightarrow{\hat{\nu}} Y(\pi, \beta) \xrightarrow{p} BO$, where p is the obvious map and $\hat{\nu}$ is a lift of $f \times s: M \longrightarrow B\pi \times BSO$.

3.7. Assume that M is a non-orientable manifold with fundamental group π and $w_2(\widetilde{M}) \neq 0$. Then $w_1(M) = f^*(\alpha)$ for some cohomology class $\alpha \in H^1(B\pi; \mathbb{Z}/2)$. Let $Y(\pi, \alpha)$ be the pullback of $\alpha: B\pi \longrightarrow K(\mathbb{Z}/2, 1)$ and the first Stiefel-Whitney class $w_1: BO \longrightarrow K(\mathbb{Z}/2, 1)$. Then ν can be factored in the form $M \xrightarrow{\hat{\nu}} Y(\pi, \gamma) \xrightarrow{p} BO$, where p is the obvious map and $\hat{\nu}$ is a lift of $f \times \nu: M \longrightarrow B\pi \times BO$.

It is easy to check the properties (1) and (2) for these factorizations. The only point worth mentioning is that the condition $w_2(\widetilde{M}) \neq 0$ guarantees that $\nu: M \longrightarrow BO$ induces a surjection on π_2. Similarly, the condition $w_1(M) \neq 0$ is equivalent to $\nu: M \longrightarrow BO$ inducing a surjection on π_1. We note that for $B = B\pi \times \mathrm{BSpin}$ (resp. $B = B\pi \times BSO$) the bordism group $\Omega_n(B)$ can be identified with the spin bordism group $\Omega_n^{\mathrm{spin}}(B\pi)$ (resp. with the oriented bordism group $\Omega_n(B\pi)$). Similarly, if π splits as $\pi = \pi' \times \mathbb{Z}/2$,

$\pi' = \ker w_1$, and $B = Y(\pi, \gamma)$ as in (3.7), then $B = B\pi' \times BO$ and the bordism group $\Omega_n(B)$ can be identified with the unoriented bordism group $\mathfrak{N}_n(B\pi')$.

Finally, if one considers manifolds with $\pi = \mathbb{Z}/2$, with $w_1(M) \neq 0$ and with $w_2(\widetilde{M}) = 0$, then $B = \mathrm{BPin}$ in the case $w_2(M) = 0$ (this is the case where M admits a Pin structure on the normal bundle and a Pin$'$ structure on the tangent bundle, see [G]) and $B = \mathrm{BPin}$ in the case $w_2(M) \neq 0$ (this is the case where $w_2(M)$ is pulled back from the unique non-zero class in $H^2(B\mathbb{Z}/2; \mathbb{Z}/2)$, hence $w_2(M) = w_1(M)^2$ and M admits a Pin structure on the tangent bundle and a Pin$'$ structure on the normal bundle, again see [G]). In the first of these cases, the bordism group $\Omega_n(B)$ is $\Omega_n^{\mathrm{pin}'}$; in the second case it is Ω_n^{pin}.

Theorem 3.3 generalizes the known results on the bordism invariance of positive scalar curvature [GL2, Thm. B, Thm. C], [R2, Thm. 2.2, Thm. 2.13], [KwSc, Prop. 1.1 and p. 283]. The proof of theorem 3.3 is exactly along the lines of [R2, Thm. 2.2], which deals with the special case where B is as in 3.4. The only point worth mentioning is the following: Given a bordism class in $Pos_n(B)$ with $n \geq 4$ we can always find a representative $\hat{\nu}: M \longrightarrow B$ such that M has a metric of positive scalar curvature and $\hat{\nu}$ is a 2-equivalence. The latter condition can always be achieved by first doing 0-surgeries on M (to make $\pi_0(M) \longrightarrow \pi_0(B)$ an isomorphism and $\pi_1(M) \longrightarrow \pi_1(B)$ surjective) and then doing 1-surgeries (to make $\pi_1(M) \longrightarrow \pi_1(B)$ an isomorphism and $\pi_2(M) \longrightarrow \pi_2(B)$ surjective). The surgery theorem 3.1 garantees that the resulting manifold again has a metric of positive scalar curvature. In particular, in theorem 2.2 of [R2] which says that the positive scalar curvature metric on a spin manifold X_2 can be propagated to a spin manifold X_1, if they represent the same class in a suitable bordism group, the assumption on the fundamental group of X_2 is superfluous.

Remark 3.8. It is still an open question whether the bordism theorem (3.3) holds for 4-dimensional manifolds. The present proof doesn't generalize to 4-dimensional manifolds due to the same reasons that the proof of the s-cobordism theorem fails for an s-cobordism with 4-dimensional boundary. Donaldson proved that the s-cobordism theorem is in fact false in this dimension and hence one might speculate that gauge theory methods could show that the bordism theorem is false in dimension 4, i.e. that there are possibly new obstructions to positive scalar curvature metrics from gauge theory in dimension 4.

The question whether the bordism theorem holds in dimension 4 can be reformulated as follows: Let $(M, \hat{\nu})$ be a B-manifold as in the bordism theorem; i.e. $\hat{\nu}$ is a 2-equivalence and $(M, \hat{\nu})$ is B-bordant to another B-manifold $(M', \hat{\nu}')$ which carries a metric of positive scalar curvature. By our discussion above we can assume that also $\hat{\nu}'$ is a 2-equivalence. This implies by a result of Kreck [Kr] that M and M' are stably diffeomorphic, i.e. for suitable positive integers s, t the connected sum $M\#s(S^2 \times S^2)$ of M with s copies of $S^2 \times S^2$ is diffeomorphic to $M'\#t(S^2 \times S^2)$. By corollary 3.2 the latter manifold carries a metric of positive scalar curvature so that the question whether M carries a metric of positive scalar curvature boils down to the following problem:

Problem 3.9. *Suppose that M is a closed 4-manifold such that $M\#(S^2 \times S^2)$ carries a metric of positive scalar curvature. Does M admit a metric of positive scalar curvature?*

Now we turn to the consequences of the bordism theorem. The theorem and the subsequent discussion show that the question whether an n-manifold M has a positive scalar curvature metric can be studied by determining the subgroup $Pos_n(B(M))$ of $\Omega_n(B(M))$. If M is simply connected and non-spin then $B(M) = BSO$ and hence $\Omega_n(B(M))$ is just Ω_n^{SO}, the bordism group of oriented n-manifolds. This bordism group has been computed by Wall [Wa]. Moreover, one knows explicit generators for Ω_n^{SO}, which admit positive scalar curvature metrics. Hence $Pos_n(B(M))$ is equal to $\Omega_n(B(M))$ in this case and the bordism theorem implies:

Corollary 3.10 [GL2, Cor. C]. *Every closed simply-connected n-manifold, $n \geq 5$, which is not spin, carries a metric of positive scalar curvature.*

As discussed in the previous section, a necessary condition for the existence of a positive scalar curvature metric on a spin manifold M is the vanishing of the index obstruction $\alpha(M) \in KO_n$.

Gromov-Lawson Conjecture (simply-connected case). *A closed simply-connected spin manifold M of dimension $n \geq 5$ carries a metric of positive scalar curvature if and only if $\alpha(M) = 0$.*

This conjecture was proved recently by the second author using techniques from stable homotopy theory. We present an outline of the proof in the next section. Previously, Gromov-Lawson had proved that the conjecture is true 'rationally'; i.e. if M is a 1-connected spin manifold with $\alpha(M) = 0$ then there is a k such that the connected sum of k copies of M carries a metric of positive scalar curvature [GL2, Cor. B]. This was improved by Miyazaki who showed that one can choose $k = 4$ [Miy2]. Moreover, Rosenberg proved that the conjecture is true for $n \leq 23$ [R3, Thm. 1.1].

The proofs of these partial results – as well as the proof of the conjecture – of course use the bordism theorem. We note that if M is simply connected and spin then $B(M) = \mathrm{BSpin}$ and hence $\Omega_n(B(M))$ is the spin bordism group Ω_n^{spin}. Thus to prove the conjecture, it suffices to find spin manifolds with positive scalar curvature metrics whose bordism classes generate the kernel of $\alpha\colon \Omega_n^{\mathrm{spin}} \to KO_n$. The spin bordism groups have been computed by Anderson, Brown and Peterson [ABP1], but in contrast to the oriented bordism groups one does not know explicit spin manifolds which are generators of Ω_n^{spin} for general n. Gromov-Lawson (resp. Miyazaki resp. Rosenberg) found spin manifolds with positive scalar curvature metrics whose bordism classes generate $ker\,\alpha \otimes \mathbb{Q}$ (resp. a subgroup of $ker\,\alpha$ containing $4\,ker\,\alpha$ resp. $ker\,\alpha$ for $n \leq 23$), which implies the results mentioned above.

Remark 3.11. Prior to Anderson-Brown-Peterson [ABP1] the spin bordism groups Ω_n^{spin} were determined for $n \leq 23$, but it was an open question whether there is a 24-dimensional spin manifold M such that the Stiefel-Whitney number $w_4 w_6 w_7^2$ is non-trivial [Mil]. Anderson-Brown-Peterson gave a positive answer to that question, but their homotopy theoretic methods do *not* give an explicit construction of such a manifold M. Later such a construction was found [Man], but to find explicit representatives for *all* generators of the spin bordism ring seems a very difficult problem.

4. Proof of the Gromov-Lawson Conjecture in the Simply Connected Case

In this section we describe the main ideas from the proof of the Gromov-Lawson conjecture [St1], [St2]. As shown in the last section it suffices to show that the kernel of $\alpha\colon \Omega_n^{\text{spin}} \longrightarrow KO_n$ is equal to the subgroup Pos_n consisting of bordism classes represented by spin manifolds with positive scalar curvature metrics (in section 2 we saw that Pos_n is contained in $ker\,\alpha$). We construct elements in Pos_n using the following remark.

Observation 4.1. *Let F be a compact Riemannian manifold of positive scalar curvature on which the group G acts by isometries. Let B be a compact manifold and let $\pi\colon E \longrightarrow B$ be a fibre bundle with fibre F and structure group G. Then E has a metric of positive scalar curvature.*

This follows easily from the O'Neill formulas for scalar curvature (e.g. [Be, 9.70d]) using a certain metric on E with totally geodesic fibres [Be, 9.59] and then shrinking the fibres.

Remark 4.2. Let $E \longrightarrow B$ be a fibre bundle over a compact manifold B whose fibre is a compact manifold F. It is *not* true that a positive scalar curvature metric on F or B implies that E has such a metric. The following is a counter example: Let Σ^9 be the exotic homotopy sphere of dimension 9 with $\alpha(\Sigma^9) \neq 0$. Then the connected sum $S^7 \times S^2 \# \Sigma^9$ has non-trivial α-invariant and hence does not admit a metric of positive scalar curvature. On the other hand Hitchin showed that $S^7 \times S^2 \# \Sigma^9$ is the total space of a fibre bundle over S^2 with fibre S^7 [Hi, p. 45].

The observation (4.1) is well known to the experts; in fact, this argument is used in [GL2], [Miy2] and [R3] to show that manifolds they construct to prove partial results towards the Gromov-Lawson conjecture admit positive scalar curvature metrics. The new idea to overcome the difficulty mentioned in remark 3.11 is to consider simultaneously *all* manifolds which are total spaces of fibre bundles with a fixed manifold F as fibre. A good choice for F is the quaternionic projective plane $\mathbb{HP}^2$, since its bordism class generates the kernel of α in dimension 8, which is the first dimension in which $ker\, \alpha$ is non-trivial. The standard riemannian metric on $\mathbb{HP}^2$ has positive scalar curvature and the projective symplectic group $G = PSp(3)$ acts on $\mathbb{HP}^2$ by isometries. The isotropy subgroups of this action are the conjugates of $H = P(Sp(2) \times Sp(1)) \subset G$. Given a manifold N and a map $f: N \to BG$ let $\hat{N} \to N$ be the pull back of the fibre bundle

$$\mathbb{HP}^2 = G/H \to BH \xrightarrow{\pi} BG. \tag{4.3}$$

via f. The bundle $\hat{N} \to N$ satisfies the assumptions of (4.1) and hence $\hat{N}$ has a positive scalar curvature metric. Moreover, if N is spin then $\hat{N}$ is again a spin manifold. We can hence define a homomorphism

$$\Psi: \Omega^{\mathrm{spin}}_{n-8}(BG) \longrightarrow \Omega^{\mathrm{spin}}_n$$

by mapping the bordism class of $f: N \to BG$ to the bordism class of $\hat{N}$. The image of Ψ is contained in Pos_n by (4.1).

Theorem 4.4. *Let $[M]$ be a bordism class in the kernel of $\alpha: \Omega^{\mathrm{spin}}_n \to KO_n(pt)$. Then there is an odd number r such that $r[M]$ is in the image of Ψ.*

It is possible to strengthen this result: one can always choose $r = 1$ [KrSt]. This stronger result implies $ker\, \alpha = Pos_n$ and hence the Gromov-Lawson conjecture. Alternatively, the theorem above shows that $r\, ker\, \alpha \subset Pos_n$ for some odd integer r. In conjunction with Miyazaki's result $4\, ker\, \alpha \subset Pos_n$ [Miy2] (cf. §3) this implies $ker\, \alpha = Pos_n$.

Theorem 4.4 is proved by translating the statement into stable homotopy theory via the Pontrjagin-Thom construction and then using Adams spectral sequence techniques. Recall that Ω^{spin}_n is canonically isomorphic to $\pi_n(\mathrm{MSpin})$ where MSpin is the Thom spectrum associated to the projection map BSpin $\to BO$. More generally, for any space X the bordism group $\Omega^{\mathrm{spin}}_n(X)$ can be identified with $\pi_n(\mathrm{MSpin} \wedge X_+)$, where X_+ is the union of X and a disjoint basepoint. In particular, $\Omega^{\mathrm{spin}}_{n-8}(BG)$ can be identified with $\pi_n(\mathrm{MSpin} \wedge \Sigma^8 BG_+)$. Using results of Boardman [Bo, Thm. 6.20] there is a map $T: \mathrm{MSpin} \wedge \Sigma^8 BG_+ \to \mathrm{MSpin}$ such that the induced map

$$T_*: \pi_n(\mathrm{MSpin} \wedge \Sigma^8 BG_+) \to \pi_n(\mathrm{MSpin})$$

can be identified with Ψ (up to sign).

Recall from §2 that there is a map $D\colon \mathrm{MSpin} \to bo$ whose induced map on homotopy groups can be identified with α. It follows from [ABP1] that D_* is surjective after localizing at 2 (i.e. after taking the tensor product of those homotopy groups with $\mathbb{Z}_{(2)} = \{\frac{a}{b} \in \mathbb{Q} : b \text{ is prime to } 2\}$). In particular, if we denote by $\widehat{\mathrm{MSpin}}$ the fibre of D, the long exact homotopy sequence shows that $\ker \alpha \otimes \mathbb{Z}_{(2)}$ is isomorphic to $\pi_n(\widehat{\mathrm{MSpin}}) \otimes \mathbb{Z}_{(2)}$.

Note that the composition DT induces the trivial map on homotopy groups, since the image of an element $[N, f] \in \Omega_{n-8}^{\mathrm{spin}}(BG) \cong \pi_n(\mathrm{MSpin} \wedge \Sigma^8 BG_+)$ under the map $(DT)_*$ can be interpreted as $\alpha(\hat{N})$. But this is zero due to the Lichnerowicz-Hitchin theorem (2.1) since $\hat{N}$ carries a positive scalar curvature metric. Applying the family index theorem to the fibre bundle (4.3) and using the Weitzenböck formula in each fibre, we can strengthen that result to show:

Proposition 4.5 [St2]. *The composition DT is zero homotopic.*

In particular, $T\colon \mathrm{MSpin} \wedge \Sigma^8 BG_+ \to \mathrm{MSpin}$ factors through a map $\widehat{T}\colon \mathrm{MSpin} \wedge \Sigma^8 BG_+ \to \widehat{\mathrm{MSpin}}$. The following theorem is then a homotopy theoretic reformulation of theorem 4.4.

Theorem 4.6. *The homomorphism $\widehat{T}_*\colon \pi_n(MSpin \wedge \Sigma^8 BG_+) \otimes \mathbb{Z}_{(2)} \to \pi_n(\widehat{MSpin}) \otimes \mathbb{Z}_{(2)}$ is surjective.*

This result is proved using Adams spectral sequence techniques. Recall that the $\mathbb{Z}/p$-homology H_*X of a spectrum X is a comodule over the dual Steenrod algebra A_*. If H_*X is known as A_*-comodule one can use the mod p Adams spectral sequence

$$\mathrm{Ext}_{A_*}^{s,t}(\mathbb{Z}/p, H_*X) \quad \Longrightarrow \quad \pi_{t-s}(X) \otimes \mathbb{Z}_{(p)}$$

to obtain information about the homotopy groups of X [Sw, 19.9 and 19.12]. From now on let $p = 2$.

Proposition 4.7. *The induced map $\widehat{T}_*\colon H_*MSpin \wedge \Sigma^8 BG_+ \to H_*\widehat{MSpin}$ is a split surjection of A_*-comodules.*

As a corollary we get that the induced map

$$\widehat{T}_*\colon \mathrm{Ext}_{A_*}^{s,t}(\mathbb{Z}/2, H_*\mathrm{MSpin} \wedge \Sigma^8 BG_+) \to \mathrm{Ext}_{A_*}^{s,t}(\mathbb{Z}/2, H_*\widehat{\mathrm{MSpin}})$$

is a surjection of the E_2-terms of the corresponding Adams spectral sequences. Note that this does *not* imply that $\widehat{T}$ induces a surjection of the E_∞-terms, since there could be non-trivial differentials in the domain spectral sequence. Fortunately this is not the case.

Proposition 4.8. *The mod 2 Adams spectral sequence of $MSpin \wedge \Sigma^8 BG_+$ collapses.*

Hence $\widehat{T}$ induces a surjection of the E_∞-terms which implies theorem 4.6. To prove (4.8) we show that $H_*\mathrm{MSpin} \wedge \Sigma^8 BG_+$ is isomorphic to an extended $A(1)_*$-comodule $A_* \square_{A(1)_*} M$, where $A(1)_*$ is the dual of $A(1)$, the subalgebra of the Steenrod algebra generated by Sq^1 and Sq^2, and M is an $A(1)_*$-comodule. Then

$$\mathrm{Ext}^{s,t}_{A_*}(\mathbb{Z}/2, H_*\mathrm{MSpin} \wedge \Sigma^8 BG_+) \cong \mathrm{Ext}^{s,t}_{A(1)_*}(\mathbb{Z}/2, M)$$

by [Sw, Prop. 20.16]. Moreover, M is a direct sum of $A(1)_*$-comodules whose Ext-groups are known [AP, §3]. Let Y be the spectrum obtained from MSpin $\wedge\, \Sigma^8 BG_+$ by splitting off the Eilenberg-MacLane spectrum corresponding to the free summands of M [Mar]. Inspecting the E_2-term of the mod 2 Adams spectral sequence of Y we conclude that all differentials are zero for dimensional reasons and due to the multiplicative structure. This implies proposition 4.8.

For the proof of proposition 4.7 we first show using results of D. Pengelley [Pe] that $H_*\widehat{\mathrm{MSpin}} \cong A_* \square_{A(1)_*} N$, where N is a certain $A(1)_*$-comodule. Moreover, $\widehat{T}_*$ is induced by a map of $A(1)_*$-comodules $f\colon M \longrightarrow N$. A calculation shows that f is surjective and that f induces a surjection on the Q_0-homology groups (Q_0 is the Bockstein which acts as a differential on $A(1)_*$-comodules). It turns out that this suffices to conclude that f is a split surjection of $A(1)_*$-comodules which implies proposition 4.7.

5. Status of the Problem for Closed Manifolds with Finite Fundamental Group

In this section we study the following question: Given a closed manifold M of dimension n with finite fundamental group π, when does M admit a metric of positive scalar curvature? Based on a few partial results described below Rosenberg made the following conjecture:

Conjecture 5.1. *A closed manifold M of dimension $n \geq 5$ with finite fundamental group π admits a metric of positive scalar curvature if and only if all (KO_*-valued) index obstructions associated to Dirac operators with coefficients in flat bundles (on M and it covers) vanish.*

If M is a spin manifold we've seen in §2 that the indices of all the Dirac operators with coefficients in flat bundles can be combined to a single element $\alpha(M) \in KO_n(C^*_{\mathbb{R}}(\pi))$. Thus the Conjecture says in this case that M has a positive scalar curvature metric if and only if $\alpha(M)$ vanishes. If, on the other hand, the universal cover $\widetilde{M}$ (and *a fortiori* M) does not admit a spin structure, then there are no Dirac operators with coefficients in flat bundles defined on M or on any of its covers. Thus the Conjecture says in this case that M admits a metric of positive scalar curvature. Of course, it can happen that $\widetilde{M}$ admits a spin structure, but M doesn't. In this case it would be nice to combine all the index obstructions into a single obstruction $\alpha(M)$ (as in the case where M is spin), but no such formulation of the conjecture is known yet (cf. [R4, §3]).

What is the evidence we have for the Conjecture? First of all, to prove the conjecture for a finite group π it suffices to verify the conjecture for its Sylow subgroups, thanks to the following proposition which is an easy generalization of a result of Kwasik and Schultz [KwSc, Prop. 1.5 and Corollary 1.6]:

Proposition 5.2. *Let M be a closed manifold of dimension $n \geq 5$ with finite fundamental group π and let $M_p \to M$ be the covering corresponding to the Sylow p-subgroup of π. Then M admits a metric of positive scalar curvature if and only if M_p carries a metric of positive scalar curvature for all p.*

The following theorem gives a list of groups for which the conjecture is true. It includes, as far as we know, all the finite p-groups, for which the conjecture has been verified so far.

Theorem 5.3. *(A) Let M be an orientable manifold with finite fundamental group π. Then conjecture 5.1 is true in the following cases:*

(1) $w_2(\widetilde{M}) \neq 0$ *and π is a cyclic group* [R3, Thm. 2.14], [R4, Thm. 1.1].
(2) $w_2(\widetilde{M}) \neq 0$ *and π is the quaternion group of order 8* [R4, Thm. 1.5].
(3) $w_2(M) = 0$ *and π is an odd order cyclic group* [R3, Thm. 1.3], [KwSc, Thm. 1.8].
(4) $w_2(M) = 0$ *and* $\pi = \mathbb{Z}/2$.
(5) $w_2(M) \neq 0$, $w_2(\widetilde{M}) = 0$ *and* $\pi = \mathbb{Z}/2^r$ [Schu].

(B) Let M be a **non**-*orientable manifold with finite fundamental group π. Then conjecture 5.1 is true in the following cases:*

(6) $w_2(\widetilde{M}) \neq 0$ *and* $\pi = \mathbb{Z}/2$.
(7) $w_2(\widetilde{M}) = 0$ *and* $\pi = \mathbb{Z}/2$.

Parts (4), (6), and (7) are new results. The proof of (4) is based on the following result, which we expect to be also useful for the verification of the conjecture for spin manifolds whose fundamental groups are other finite 2-groups.

Theorem 5.4 [St3, Thm. 1.1]. *Let π be a finite 2-group and let M be a closed spin manifold of dimension $n \geq 5$ with fundamental group π such that its bordism class is in the kernel of*

$$\Omega_n^{\mathrm{spin}}(B\pi) = \pi_n(MSpin \wedge B\pi_+) \xrightarrow{D_*} bo_n(B\pi_+) .$$

Then M admits a metric of positive scalar curvature.

Note that if π is the trivial group this theorem implies the Gromov-Lawson conjecture. Recall from §4 that the main homotopy theoretic result needed for the proof of the Gromov-Lawson conjecture is that a certain map $\widehat{T}\colon \mathrm{MSpin} \wedge \Sigma^8 BG_+ \longrightarrow \widehat{\mathrm{MSpin}}$ induces a surjection on homotopy groups localized at 2. Theorem 5.4 is a corollary of the stronger result that $\widehat{T}$ is in fact a split surjection of spectra.

Proof of theorem 5.3 (4). Using the fact that the conjecture is true in the simply connected case it suffices to decide which bordism classes in $\widetilde{\Omega}_n^{\mathrm{spin}}(B\mathbb{Z}/2)$ by can be represented by positive scalar curvature manifolds. Let

$$\tilde{\alpha}\colon \Omega_n^{\mathrm{spin}}(B\mathbb{Z}/2) \longrightarrow KO_n(pt)$$

be the map which sends a bordism class $[M, f]$ to $\alpha(\widetilde{M})$, where $\widetilde{M}$ is the pull back of the double covering $E\mathbb{Z}/2 \longrightarrow B\mathbb{Z}/2$ via f ($\widetilde{M}$ is the universal covering of M if f is 2-connected). If M has a metric of positive scalar curvature then then so does $\widetilde{M}$ and hence the bordism class $[M, f]$ is in the kernel of $\tilde{\alpha}$. We claim that the converse is true, too, which implies the conjecture in the case at hand.

We note that $\tilde{\alpha}$ factors in the form

$$\widetilde{\Omega}_n^{\mathrm{spin}}(B\mathbb{Z}/2) \xrightarrow{D_*} bo_n(B\mathbb{Z}/2) \xrightarrow{\hat{\alpha}} KO_n(pt) .$$

Hence theorem 5.4 shows that it is in fact enough to represent the elements in the kernel of $\hat{\alpha}$ by positive scalar curvature manifolds. The groups $bo_n(B\mathbb{Z}/2)$ have been computed by Mahowald [Mah, Lemma 7.3]:

$$bo_n(B\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & n \equiv 1,2 \mod 8,\ n \geq 0 \\ \mathbb{Z}/2^{2k} & n = 4k-1,\ k \equiv 0 \mod 2 \\ \mathbb{Z}/2^{2k+1} & n = 4k-1,\ k \equiv 1 \mod 2 \\ 0 & \text{otherwise} \end{cases}$$

For $n \equiv 1,2,3,7 \mod 8$, $n > 0$ let $g_n \in bo_n(B\mathbb{Z}/2)$ be the following element:

$$g_n = \begin{cases} D_*([S^1, i])\, v^{4k} & n = 8k+1 \\ D_*([S^1, i])\, \eta\, v^{4k} & n = 8k+2 \\ D_*([\mathbb{RP}^{4k-1}, i]) & n = 4k-1 \end{cases}$$

Here i is the obvious inclusion into $B\mathbb{Z}/2 = \mathbb{RP}^\infty$ and we consider S^1 (resp. $\mathbb{RP}^{4k-1}$) as spin manifolds. There are exactly two choices of spin structures on these manifolds and we pick the one which is trivial in the sense that it extends to a spin structure on D^2 (resp. the disk bundle of $H \otimes H$, where H is the Hopf line bundle over $\mathbb{CP}^\infty$ and we identify $\mathbb{RP}^{4k-1}$ with the sphere bundle of $H \otimes H$). Furthermore, $\eta \in bo_1(S^0) \cong \mathbb{Z}/2$ is the non-trivial element and $v \in bo_8(S^0)$ is the periodicity element. We can multiply by these elements, since bo is a ring spectrum which makes $bo_*(X)$ a module over $bo_*(S^0)$ for any X.

Claim. g_n is a generator of $bo_n(B\mathbb{Z}/2)$.

We note that the image of an element $D_*([M,f]) \in \widetilde{\Omega}_n^{\mathrm{spin}}(X)$ under the natural transformation $h: bo_n(X) \longrightarrow H_n(X; \mathbb{Z}/2)$ is just $f_*([M])$, where $[M]$ is the $\mathbb{Z}/2$-fundamental class. In particular, $h(g_n) \neq 0$ for $n = 1$ and $n = 4k-1$. This shows that in these degrees g_n is an element of Adams filtration zero and hence a generator. For $n \equiv 1, 2 \mod 8$, $n > 1$ the structure of $bo_*(B\mathbb{Z}/2)$ as module over $bo_*(S^0)$ implies that g_n is a generator (the module structure can be read off from the multiplicative structure of the E_2-term of the Adams spectral sequence converging to $bo_*(B\mathbb{Z}/2)$).

Claim. For $n \equiv 1, 2 \mod 8$ we have $\hat{\alpha}(g_n) \neq 0$.

Let $\widetilde{S}^1$ be the total space of the pull back of the double covering $E\mathbb{Z}/2 \longrightarrow B\mathbb{Z}/2$ via $i: S^1 \longrightarrow B\mathbb{Z}/2$. It is an amusing exercise to show that the lift of the trivial spin structure on S^1 is the *non-trivial* spin structure on $\widetilde{S}^1$. In particular, $\hat{\alpha}(g_1) = \tilde{\alpha}([S^1, i]) = \alpha(\widetilde{S}^1)$ is non-zero. We note that $\hat{\alpha}$ is a bo_*-module map, since if we identify $KO_n(pt)$ with $bo_n(S^0)$ for $n \geq 0$ then $\hat{\alpha}$ is the map of bo-homology groups induced by the Kahn-Priddy transfer [Bo, Thm. 6.20]. This shows that $\hat{\alpha}(g_{8k+1}) = v^k \hat{\alpha}(g_1) = v^k \eta$ (resp. $\hat{\alpha}(g_{8k+2}) = v^k \eta \hat{\alpha}(g_1) = v^k \eta^2$) and these elements are non-trivial in $bo_*(S^0)$ by Bott periodicity.

This shows that the elements g_{4k-1}, which are represented by the positive scalar curvature manifolds $\mathbb{RP}^{4k-1}$ generate the kernel of $\hat{\alpha}$ and hence proves the theorem. □

Proof of theorem 5.3 (6, 7). Case (6) of theorem 5.3 follows easily from the analysis in example 3.7 above. As we pointed out there, we only need to show that each class in the unoriented bordism group $\mathfrak{N}_n$ ($n > 0$) can be represented by a manifold of positive scalar curvature. As remarked in the proof of [R4, Theorem 1.2], this is possible since $\mathfrak{N}_*$ is a polynomial algebra over the field $\mathbb{F}_2$ of two elements, with generators represented by even-dimensional real projective spaces and by hypersurfaces of degree (1, 1) in products of pairs of real projective spaces.

For case (7), we need to deal similarly with those generators of Ω_n^{pin} and $\Omega_n^{\mathrm{pin}'}$ for which certain obstructions vanish. The analysis is somewhat complicated and will be deferred in part to a future paper of the authors, but we can at least give a sketch. First, an analogue of Theorem 5.4 is used to show that one has positive scalar curvature on all generators of Ω_n^{pin} and $\Omega_n^{\mathrm{pin}'}$ except for those associated to summands in $MPin$ and $MPin'$ coming from bo. The relevant groups are then computed in [ABP2] and in [G]. In the case of $\Omega_*^{\mathrm{pin}} \xrightarrow[\gamma]{\cong} \widetilde{\Omega}_{*+1}^{\mathrm{spin}}(\mathbb{RP}^\infty)$, $bo \wedge \mathbb{RP}^\infty$ contributes big cyclic summands in dimensions 2 (mod 4) and $\mathbb{Z}/2$'s in dimensions $0, 1$ (mod 8). By [G, Corollary 3.5], the former are generated by $\mathbb{RP}^{4k+2}$, and so carry positive scalar curvature. The $\mathbb{Z}/2$ summands in dimensions $0, 1$ (mod 8) do **not** carry positive scalar curvature because of an index obstruction. For if M^n is the Pin manifold with spin double cover $\widetilde{M}$ representing one of these summands, $\gamma(M) \in \widetilde{\Omega}_{n+1}^{\mathrm{spin}}(\mathbb{RP}^\infty)$ is represented by $Y = (\widetilde{M} \times S^1)/(\tau \times c)$, where τ is the involution on $\widetilde{M}$ with quotient M, and where c is complex conjugation, a map of degree -1, on S^1. If M were to have a metric of positive scalar curvature, we could lift it to $\widetilde{M}$, take the product with the standard metric on S^1, and get a metric of positive scalar curvature on Y. However, the $\mathbb{Z}/2$ summands in Ω_n^{pin} dimensions $n \equiv 0, 1$ (mod 8) map under γ to the classes in $\widetilde{\Omega}_{n+1}^{\mathrm{spin}}(\mathbb{RP}^\infty)$ for which the $\mathbb{Z}/2$ index of the twisted Dirac operator is non-zero, so Y cannot admit positive scalar curvature.

In the Pin$'$ case, the relevant summand in the bordism group has big cyclic summands in dimensions 0 (mod 4) and $\mathbb{Z}/2$'s in dimensions $2, 3$ (mod 8) [G, §2]. Again, by [G, Corollary 3.5], the former are generated by $\mathbb{RP}^{4k}$, and so carry positive scalar curvature. The $\mathbb{Z}/2$ summands in dimensions $2, 3$ (mod 8) do **not** carry positive scalar curvature because of an index obstruction. In dimension 2 (mod 8) this is clear from the exact sequence of [G, Theorem 3.1], since the double cover $\widetilde{M}$ of a Pin$'$ manifold M in the relevant class is a spin manifold with $\alpha(\widetilde{M}) \neq 0$. A different argument is needed in dimension 3 (mod 8), using the fact that the $\mathbb{Z}/2$ summand here maps non-trivially to bo_2. □

It seems that the problems one encounters when trying to prove conjecture 5.1 for more groups are all related and come from the fact that there is no 1-dimensional manifold with positive scalar curvature. For example, the projection $\mathbb{Z}^n \longrightarrow (\mathbb{Z}/q)^n$ induces a map f from the n-dimensional torus T^n to $B(\mathbb{Z}/q)^n$.

Question 5.5. Can the bordism class $[T^n, f] \in \Omega_n^{\mathrm{spin}}(B(\mathbb{Z}/q)^n)$ be represented by a positive scalar curvature manifold?

If conjecture 5.1 is true the answer should be positive. On the other hand, if we replace $\mathbb{Z}/q$ by the infinite cyclic group $\mathbb{Z}$ the answer is negative, since

f is a 2-equivalence in this case and hence by the bordism theorem 3.3 a positive answer would imply that T^n admits a metric of positive scalar curvature which contradicts results of [GL1, Cor. B].

Let L be a complex line bundle over the Kummer surface K whose first Chern class is not divisible (K is a simply connected 4-dimensional spin manifold with $\hat{A}(K) = 2$). Let M be the sphere bundle of L. Then M is a simply connected spin manifold with a free S^1-action. It follows from 5.3 (3,4) that M does have a metric of positive scalar curvature which is $\mathbb{Z}/q$-equivariant if q is odd or if $q = 2$. There is no S^1-equivariant metric of positive scalar curvature on M [BB, Theorem C], but the open question is:

Question 5.6. Does M admit a $\mathbb{Z}/2^r$-equivariant metric of positive scalar curvature?

6. Status of the Problem for Closed Manifolds with Certain Infinite Fundamental Groups

While the positive scalar curvature problem does not in its original formulation depend on the size of the fundamental group, the problem tends to take on a different flavor when the fundamental group is "large" compared to the "size" of the manifold. For instance, in dimension 2, the closed manifolds admitting metrics of positive scalar curvature are exactly those with finite fundamental group. In dimension 3, the Thurston geometrization conjecture (or more precisely, a special case of it, that any 3-manifold with finite fundamental group should admit a metric of constant positive **sectional** curvature) would imply that a closed orientable 3-manifold admits a metric of positive scalar curvature if and only if it has no $K(\pi, 1)$ summands in its prime decomposition ([SY1] and [GL3, Theorem 8.1]). Thus, from the point of view of 3-manifolds, finite groups and $\mathbb{Z}$ are not "large" fundamental groups, but the fundamental group of a 3-dimensional $K(\pi, 1)$ manifold is. Also, the tools needed for studying the problem are somewhat different in the "opposite" cases of finite and "large" fundamental group. This was already noted by Gromov and Lawson in the differences of approach in their two original papers [GL1] and [GL2], though the C*-algebraic Dirac operator approach to the problem (Theorem 2.2) doesn't depend on any assumption about the size of π_1.

When the fundamental group of a closed manifold is infinite, its universal cover is non-compact, and one can try to use the methods of §7 below to find obstructions to existence of a complete metric of uniformly bounded positive scalar curvature on the universal cover. On the other hand, when the fundamental group is infinite, it may not have many finite-dimensional representations, so merely studying the Dirac operator on finite-dimensional flat vector bundles may not give much information. Furthermore, we know from the example of $\mathbb{CP}^2 \# T^4$ quoted above in §2 that the analogue of

Conjecture 5.1 cannot hold in general, and it is not known exactly how the conjecture should be modified. (The example has a non-spin universal cover, but does not admit a metric of positive scalar curvature.)

The main conjecture with regard to "large" fundamental group is:

Conjecture 6.1 (Gromov-Lawson). *No closed aspherical manifold M admits a metric of positive scalar curvature.*

This is known in an enormous number of cases, basically for all aspherical manifolds with good geometrical properties. As we remarked in §2, it would follow from the strong Novikov conjecture (see [R1, §3B]), even without assuming that M has a spin structure, and hence, by results of Kasparov [K], when $\pi_1(M)$ is the fundamental group of a complete manifold of non-positive curvature or is a discrete subgroup of a connected Lie group. By more recent results of Kasparov and Skandalis [KaSk], the strong Novikov conjecture, and hence Conjecture 6.1, also holds if $\pi_1(M)$ can be embedded in $GL(n)$ of a number field. The conjecture is proved with various other geometrical hypotheses in [GL3], [Mo], [CM], [CGM]. Sample sufficient conditions on M are enlargeability or Λ^2-enlargeability [GL3]—the definitions of these are technical, but they are also satisfied in many of the cases treated by Kasparov—or that $\pi_1(M)$ be a "hyperbolic" or "semi-hyperbolic" group in the combinatorial sense [CGM].

As for results on infinite fundamental groups not related to Conjecture 6.1, the obstruction theorem (Theorem 2.2), the bordism theorem (Theorem 3.3), and the Gromov-Lawson Conjecture (Theorem 4.4) make it possible to obtain fairly complete results about positive scalar curvature on manifolds for which the fundamental group is infinite but still "not too big," for instance free abelian with rank smaller than the dimension. (The relevant property is that the homological dimension of the fundamental group is smaller than the dimension of the manifolds considered.) We give a number of sample results, without trying to give an exhaustive list of all theorems along these lines.

Theorem 6.2. *Suppose M^n is a closed oriented n-manifold with fundamental group π and universal cover $\widetilde{M}$, with $n \geq 5$. Then:*

(1) *If $w_2(\widetilde{M}) \neq 0$ and π is a free group or $\pi = \pi_1(S_g^2)$, the fundamental group of a closed orientable surface S_g of genus g, or if π is free abelian of rank $k < n$, then M admits a metric of positive scalar curvature.*

(2) *If $w_2(M) = 0$ and s is a spin structure on M, and if π is free or $\pi = \pi_1(S_g^2)$, or if π is free abelian of* **any** *rank k, then M admits a metric of positive scalar curvature if and only if $D_*([M, s, f]) = 0$ in $KO_n(B\pi)$. Here $M \xrightarrow{f} B\pi$ is the classifying map for the universal cover $\widetilde{M} \to M$ of M.*

Proof. The free group case of (1) is in [Miy1, Theorem 5.9], the free abelian case of (1) is in [Miy1, Theorem 5.6], and the "only if" part of (2), as well as the "if" part with $n \leq 23$, are in [R3, Theorems 3.5, 3.6, and 3.8]. For the rest of (2), note that the proofs of [R3, Theorems 3.5, 3.6, and 3.8] go through for all $n \geq 5$ once one has the Gromov-Lawson Conjecture in the simply connected case. Furthermore we observe that the rank restriction $n \geq k$ in [R3, Thm. 3.6] is superfluous. The other cases of (1) are similar. □

Note that what we are relying on in the proof above is the property of the group π that $B\pi$ splits stably as a wedge of spheres. (This also holds for many 3-manifold groups, so the same method will work for them as well.) Then for any bordism theory Ω^B, generators of $\Omega^B_*(B\pi)$ may be obtained from products of generators of Ω^B_* with a point, with a sphere, with a torus, or with a closed surface, and its easy to check when the positive scalar curvature condition holds.

The difference between cases (1) and (2) in the free abelian case is due to the fact that when $w_2(\widetilde{M}) \neq 0$, there are no obvious index-theoretic obstructions to positive scalar curvature, so we have to check for positive scalar curvature on manifolds in the same bordism class as T^n. This manifold does **not** admit positive scalar curvature, but it is not clear what happens when we replace it by a non-spin manifold in the same bordism class. A reasonable conjecture, supported by the results of [GL3], is that if $w_2(\widetilde{M}) \neq 0$ and $\pi = \mathbb{Z}^k$, then M^n admits a positive scalar curvature metric if and only if $f_*([M]) \in H_n(B\pi; \mathbb{Z})$ vanishes. Here $[M]$ is the fundamental class of M, and as above, f is the classifying map for the universal covering.

7. Complete Manifolds

It is known that any non-compact manifold (of dimension > 1) admits a Riemannian metric with positive **sectional** curvature [Gr, Theorem 4.5.1], hence certainly with positive scalar curvature. The interesting questions about positive scalar curvature on non-compact manifolds are therefore whether one can find a **complete** metric of positive scalar curvature (perhaps in a specified quasi-isometry class), and whether one can do this bounding the scalar curvature away from 0 and/or ∞. The most important results on this problem to date may be found in [GL3], though much of the recent work on index theory for open manifolds, such as [Roe1], [Roe2], and [Roe3], also has implications for this question.

An interesting test case for the problem is provided by manifolds of the form $X^n \times \mathbb{R}^k$, where X^n is a closed manifold which does **not** admit a metric of positive scalar curvature. Obviously, $\mathbb{R}$ has no metrics of positive scalar curvature. It was observed in [GL3] that $\mathbb{R}^2$ has complete metrics of positive scalar curvature (such as the obvious metric on a paraboloid of revolution), but no complete metrics with scalar curvature bounded below by a positive constant. (In dimension 2, scalar curvature is essentially the same as Ricci curvature, and by Myers's Theorem, any complete manifold with uniformly positive Ricci curvature is compact.) However, $\mathbb{R}^k$ has complete metrics of uniformly positive scalar curvature for $k \geq 3$ (take the Riemannian product of the constant curvature metric on S^{k-1} with the usual metric on $[0, \infty)$, then glue on a suitable metric on D^k). A reasonable conjecture is that if X^n is a closed manifold which does not admit a metric of positive scalar curvature, then these results persist for $X^n \times \mathbb{R}^k$, i.e., one has:

Conjecture 7.1. *Let X^n be a closed manifold which does not admit a metric of positive scalar curvature. Then*

(1) *$X^n \times \mathbb{R}$ does not admit a complete metric of positive scalar curvature; and*
(2) *$X^n \times \mathbb{R}^2$ admits no complete metrics of uniformly positive scalar curvature.*

This conjecture is augmented by the following facts:

Proposition 7.2. *Let X^n be a closed manifold which does not admit a metric of positive scalar curvature. Then*

(2′) *$X^n \times \mathbb{R}^2$ admits complete metrics of (non-uniformly) positive scalar curvature; and*
(3) *$X^n \times \mathbb{R}^k$ admits complete metrics of uniformly positive scalar curvature when $k \geq 3$.*

Obviously, taking the product of any metric on X with a rescaled version of the above metrics on $\mathbb{R}^k$ proves (3). (1) and (2) of the conjecture (with an additional technical condition for (2)) are proved in [GL3, Corollary 6.13 and Theorem 7.5] when X is enlargeable. Various other cases of part (1) of the conjecture can be proved using the minimal surface technique. Suppose for simplicity that X is oriented. If $Y^{n+1} = X^n \times \mathbb{R}$ has a metric of positive scalar curvature and there is a stable minimal hypersurface M^n in the homology class defined by $[X]$ in $H_n(Y; \mathbb{Z})$, then M is cobordant to X (for some $t \in \mathbb{R}$, $X \times \{t\}$ and M are disjoint and are the two boundary components of some "segment" in Y). By the basic minimal surface argument, M has a metric of positive scalar curvature, and under suitable conditions, the cobordism can be used to "propagate" positive scalar curvature to M via Theorem 3.3.

Some cases of (2′) are proved in [GL3], and one can prove this in general by taking a warped product metric, using a suitable metric of positive scalar curvature on $\mathbb{R}^2$ and function $f : \mathbb{R}^2 \to \mathbb{R}^+$. More precisely, fix a metric on X and give $\mathbb{R}^2$ the metric of a surface of revolution, described by rotating around the x-axis the graph $(x, y) = (g(r), h(r))$, $r \geq 0$, where for convenience $g(0) = h(0) = 0$, $g'(0) = 1$, $(g')^2 + (h')^2 \equiv 1$, and $h''(r) < 0$ for

$r > 0$. Here r denotes the geodesic distance in the surface from the origin and the Gaussian curvature of the metric is given by $K = -h''/h$. We can adjust h so that $K(0)$ is any desired positive constant and K decreases as $O(r^{-2})$. Take $f = r$ (smoothed out near $r = 0$). Then one can see (using the formula for the radial part of the Laplace-Beltrami operator) that $\|\nabla f\|^2/f^2$ and $\Delta f/f$ both decay like a constant times r^{-2}. If we give $X^n \times \mathbb{R}^2$ the warped product metric $f^2dx^2 + ds^2$, where dx^2 is a metric on X, then by [GL3, (7.35)], the scalar curvature of $X^n \times \mathbb{R}^2$ is

$$\kappa = 2K + \frac{\kappa_X}{f^2} - n(n-1)\frac{\|\nabla f\|^2}{f^2} - 2n\frac{\Delta f}{f}.$$

All terms decay as $O(r^{-2})$, and we can adjust the various parameters so that κ is positive everywhere.

This leaves (2), which it seems ought to be approachable in the spin case by versions of non-compact index theory. Completeness of the metric implies that the Dirac operator D on $X \times \mathbb{R}^2$ is essentially self-adjoint [W], and the condition of uniformly positive scalar curvature will imply that D has a bounded inverse, hence that index invariants vanish.

The question of what non-compact manifolds admit complete metrics of positive scalar curvature is still far from being understood in more general situations, and to close, we shall content ourselves with quoting a few results of Gromov and Lawson and of Roe. By [GL3, Theorems 6.13 and 8.4], manifolds which admit complete hyperbolic metrics of finite volume, and 3-manifolds which contain incompressible surfaces, do not admit any complete metrics at all with positive scalar curvature. The class of open manifolds without complete metrics of uniformly positive scalar curvature is bigger, and includes open 3-manifolds containing a "small" circle [GL3, Theorem 8.7], one with infinite order in H_1 and such that the normal circle also has infinite order in H_1. Finally, the index theory of Roe puts substantial restrictions on when one can construct a metric of positive scalar curvature or uniformly positive scalar curvature in a given quasi-isometry (or "bornotopy") class. By [Roe1, Proposition 3.3], there is no complete metric of positive scalar curvature in the strict quasi-isometry class of an infinite amenable covering of a closed spin manifold with $\hat{A} \neq 0$. By [Roe3, Proposition 6.14], if M^n is a complete spin manifold satisfying a condition that roughly says that its metric end looks like S^{n-1} (for instance, if $M = \mathbb{R}^n$ and M is in the same bornotopy class as standard Euclidean space), then the scalar curvature of M can't be uniformly positive.

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