

A “stable” version of the Gromov-Lawson conjecture

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ABSTRACT. We discuss a conjecture of Gromov and Lawson, later modified by Rosenberg, concerning the existence of positive scalar curvature metrics. It says that a closed spin manifold M of dimension $n \geq 5$ has a positive scalar curvature metric if and only if the index of a suitable “Dirac” operator in $KO_n(C^*(\pi_1(M)))$, the real K -theory of the group C^* -algebra of the fundamental group of M , vanishes. It is known that the vanishing of the index is necessary for existence of a positive scalar curvature metric on M , but this is known to be a sufficient condition only if $\pi_1(M)$ is the trivial group, $\mathbb{Z}/2$, an odd order cyclic group, or one of a fairly small class of torsion-free groups.

We note that the groups $KO_n(C^*(\pi))$ are periodic in n with period 8, whereas there is no obvious periodicity in the original geometric problem. This leads us to introduce a “stable” version of the Gromov-Lawson conjecture, which makes the weaker statement that the product of M with enough copies of the “Bott manifold” B has a positive scalar curvature metric if and only if the index of the Dirac operator on M vanishes. (Here B is a simply connected 8-manifold which represents the periodicity element in $KO_8(pt)$.) We prove the stable Gromov-Lawson conjecture for all spin manifolds with finite fundamental group and for many spin manifolds with infinite fundamental group.

Introduction

Let M be an n -dimensional manifold (all manifolds considered in this paper are smooth, compact, and, unless otherwise specified, they are connected and their boundary is empty). In this note we study the following

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QUESTION. Under which (topological) conditions does M admit a positive scalar curvature metric, i.e. a Riemannian metric whose scalar curvature function is positive everywhere?

This question has certainly a more differential geometric flavor than one would expect at a conference on homotopy theory. However, thanks to results of Gromov-Lawson and (independently) Schoen-Yau, the answer to the above question depends only on the bordism class of M in a suitable bordism group. Via the Pontrjagin-Thom construction, this bordism group can be interpreted as a homotopy group of a Thom spectrum, and then homotopy theoretic techniques, notably the Adams spectral sequence, are used to get (partial) results to the above question.

1. Constructions of positive scalar curvature metrics

Let g be a Riemannian metric on a manifold M of dimension n . The scalar curvature is a smooth function $s: M \rightarrow \mathbb{R}$, which is obtained from the curvature tensor by contracting twice. More geometrically, the scalar curvature at a point p is a measure of how fast the volume of the ball of radius r around p is growing with r . More precisely, we compare $\text{Vol } B_r(M, p)$, the volume of the ball of radius r around p , with $\text{Vol } B_r(\mathbb{R}^n, 0)$, the volume of the ball of radius r in n -dimensional Euclidean space \mathbb{R}^n , by expressing their quotient as a power series in r . We get:

$$(1.1) \quad \frac{\text{Vol } B_r(M, p)}{\text{Vol } B_r(\mathbb{R}^n, 0)} = 1 - \frac{s(p)}{6(n+2)}r^2 + \dots$$

In particular, if a Riemannian manifold has positive scalar curvature then the volume of (small) balls grows slower than the volume of the corresponding Euclidean balls. Examples of manifolds with positive scalar curvature are the n -dimensional sphere S^n , with its usual metric, as well as certain quotients of S^n , like projective spaces (real, complex, or quaternionic), equipped with the metric induced by the standard metric on S^n . Here, we assume of course $n \geq 2$, since the scalar curvature of a 1-dimensional manifold is identically zero, as one can see immediately from equation (1.1).

One possible approach to the question of the introduction is to study whether the answer changes when we modify M . A modification of manifolds which is popular among geometric topologists and has been very successful in the diffeomorphism classification of manifolds is “surgery”. Given a manifold M of dimension n , and an embedding $S^k \times D^{n-k} \hookrightarrow M$, we remove from M the open subset $S^k \times \mathring{D}^{n-k}$ to get a manifold with boundary $S^k \times S^{n-k-1}$ and glue it with $D^{k+1} \times S^{n-k-1}$ along their common boundary to get a closed manifold \widehat{M} . One says that \widehat{M} is the result of a k -surgery on M . A crucial result in the subject is the following “Surgery Theorem” proved independently by Gromov-Lawson and Schoen-Yau.

1.2 SURGERY THEOREM (GROMOV-LAWSON [GL2], SCHOEN-YAU [SY]).

Let M be an n -dimensional manifold (not necessarily connected) which admits a positive scalar curvature metric, and suppose that \widehat{M} is obtained from M by a k -surgery. If $n - k$, the “codimension” of the surgery, is greater than or equal to 3, then \widehat{M} carries a positive scalar curvature metric.

SKETCH OF PROOF. The key step in the proof of this result is a careful deformation of the original metric on $M \setminus S^k \times \dot{D}^{n-k}$ in a neighbourhood of its boundary in such a way that

- (1) The deformed metric has still positive scalar curvature.
- (2) The deformed metric fits together with the “standard metric” on $D^{k+1} \times S^{n-k-1}$ to give a metric on \widehat{M} .

Here the “standard metric” is the product of the usual metrics on D^{k+1} and on S^{n-k-1} . We observe that this metric has positive scalar curvature *provided* $n - k \geq 3$, which explains the codimension condition in the Surgery Theorem. \square

For applications of this result, it is important to characterize those manifolds obtainable from a given manifold N by a sequence of surgeries of codimension ≥ 3 . Morse theory implies that a manifold M can be obtained from N by a sequence of surgeries (without restrictions on the codimension) if and only if N is bordant to M , i.e. if there is a manifold W whose boundary is the disjoint union of M and N . The crucial ‘if’ part of this assertion can be seen as follows: Given a bordism W between M and N , we can find a “Morse function” on W , that is, a smooth function $h: W \rightarrow [0, 1]$ satisfying the following conditions:

- (1.3a) $h^{-1}(0) = M$, $h^{-1}(1) = N$.
- (1.3b) The Hessian of h at a critical point is non-degenerate.
- (1.3c) There is at most one critical point on each level set $h^{-1}(t)$, $0 < t < 1$, and none for $t = 0, 1$.

Given a subinterval $[t_1, t_2] \subseteq [0, 1]$ the level sets $h^{-1}(t_1)$ and $h^{-1}(t_2)$ are diffeomorphic, provided there is no critical point in $h^{-1}([t_1, t_2])$. If there is exactly one critical point in $h^{-1}([t_1, t_2])$, then $h^{-1}(t_1)$ is obtained from $h^{-1}(t_2)$ by a surgery whose codimension is the index of this critical point (cf. [Mi]). In particular, we get the following result which we state as a lemma for future reference.

1.4 LEMMA. *A manifold M can be obtained from another manifold N by a sequence of surgeries of codimension ≥ 3 , if we can find a bordism W between them and a Morse function $h: W \rightarrow [0, 1]$ whose critical points have index ≥ 3 .*

Gromov and Lawson noticed that one can find such a pair (W, h) , provided M and N represent the same element in a suitable bordism group. Before stating their result, we introduce some notation. Given a topological space X , we denote by $\Omega_n(X)$ the bordism classes of pairs (M^n, f) , where M is an n -dimensional closed manifold, and $f: M \rightarrow X$ is a map (two such pairs (M_1, f_1) , (M_2, f_2) are bordant if there is a bordism W between M_1 and M_2 , and a map $F: W \rightarrow X$

whose restriction to M_i is f_i for $i = 1, 2$). Disjoint union of such pairs gives $\Omega_n(X)$ the structure of an abelian group. We are actually interested in variations of this bordism group where all manifolds are equipped with compatible orientations resp. spin structures. The usual notation for these bordism groups is $\Omega_n^{SO}(X)$ resp. $\Omega_n^{spin}(X)$. Let $\text{Pos}_n^{SO}(X)$ (resp. $\text{Pos}_n^{spin}(X)$) be the subgroup of $\Omega_n^{SO}(X)$ (resp. $\Omega_n^{spin}(X)$) consisting of bordism classes represented by pairs (M, f) for which M admits a positive scalar curvature metric.

1.5 BORDISM THEOREM (GROMOV-LAWSON [GL2]). *Let M be a manifold of dimension $n \geq 5$ with fundamental group π . Let $u: M \rightarrow B\pi$ be the classifying map of the universal covering $\widetilde{M} \rightarrow M$. Then M has a positive scalar curvature metric if and only if*

$$[M, u] \in \begin{cases} \text{Pos}_n^{spin}(B\pi) & \text{if } M \text{ is spin,} \\ \text{Pos}_n^{SO}(B\pi) & \text{if } M \text{ is oriented and } \widetilde{M} \text{ is not spin.} \end{cases}$$

REMARK. This result doesn't cover non-orientable manifolds or manifolds which are non-spin, but whose universal cover is spin. There is however a general result **[RS1]** covering all manifolds of dimension $n \geq 5$, where the bordism groups $\Omega_n^{SO}(X)$ resp. $\Omega_n^{spin}(X)$ have to be replaced by more general bordism groups.

PROOF. We have to show that if (M, u) is bordant to (N, f) , and N has a positive scalar curvature metric, then so does M . Combining the Surgery Theorem with Lemma 1.4 we see that it suffices to find a bordism W between M and N and a Morse function $h: W \rightarrow [0, 1]$ whose critical points have index ≥ 3 .

We recall from Morse theory that W is homotopy equivalent to a space obtained by attaching cells to M , with each cell of dimension i corresponding to a critical point of index i of a Morse function on W . In particular, if there is a Morse function whose critical points have index ≥ 3 , then the inclusion $M \hookrightarrow W$ is a 2-equivalence (i.e. it induces an isomorphism on the i -th homotopy group for $i < 2$ and a surjection for $i = 2$). Conversely, the techniques used in the proof of the s -Cobordism Theorem show that if the dimension of M is greater or equal to 5 and the inclusion $i: M \hookrightarrow W$ is a 2-equivalence, then one can in fact find a Morse function on W whose critical points have index ≥ 3 .

Now let's assume that M is a spin manifold, and that (W, F) is a bordism between (M, u) and (N, f) . Changing the bordism (W, F) if necessary by surgeries in the interior of W , we can assume that $F: W \rightarrow B\pi$ is a 3-equivalence. Since $F \circ i = u$, and u is a 2-equivalence, this implies that i is a 2-equivalence and proves the theorem in this case.

If M and hence W are not spin, this argument doesn't work, since there might be a non-trivial class in $\pi_2(W)$ which can't be killed by surgery since an embedded 2-sphere representing it has a non-trivial normal bundle. In this case we replace u by $u' = u \times \nu_M: M \rightarrow B\pi \times BSO$, where ν_M is the classifying map of the stable normal bundle of M . The assumption that \widetilde{M} is not spin guarantees that u' is a 2-equivalence. Moreover, $u' = F' \circ i$, where $F' = F \times \nu_W: W \rightarrow$

$B\pi \times BSO$, and as before we can make F' a 3-equivalence by surgeries on W , since any element in the kernel of $F'_*: \pi_2(W) \rightarrow \pi_2(B\pi \times BSO)$ can be represented by an embedded 2-sphere with trivial normal bundle. \square

The Bordism Theorem shows that classifying all manifolds of positive scalar curvature of dimension n with fundamental group π , which are spin manifolds (resp. orientable with \widetilde{M} non-spin) is equivalent to determining the group $\text{Pos}_n^{\text{spin}}(B\pi)$ (resp. $\text{Pos}_n^{\text{SO}}(B\pi)$). The following two observations turn out to be very useful for constructing elements of $\text{Pos}_n(B\pi)$.

1.6 OBSERVATION. *If M, N are manifolds, and M admits a positive scalar curvature metric, then so does $M \times N$.*

PROOF. Let g be a positive scalar curvature metric on M and let h be any Riemannian metric on N . The scalar curvature of the product metric $g \oplus h$ on $M \times N$ is not necessarily positive. However, if we ‘shrink’ M by replacing the metric g by tg for some real number $0 < t < 1$, the scalar curvature at a point (p, q) of $M \times N$ is given by

$$s^{tg \oplus h}(p, q) = s^{tg}(p) + s^h(q) = \frac{1}{t}s^g(p) + s^h(q),$$

which is positive for t sufficiently small. Since our manifolds are compact, we can choose t such that the scalar curvature of $tg \oplus h$ is positive everywhere. \square

This observation shows in particular that $\text{Pos}_*^{\text{SO}}(pt) \stackrel{\text{def}}{=} \bigoplus_n \text{Pos}_n^{\text{SO}}(pt)$ is an ideal in the oriented bordism ring $\Omega_*^{\text{SO}}(pt) \stackrel{\text{def}}{=} \bigoplus_n \Omega_n^{\text{SO}}(pt)$ (multiplication induced by Cartesian product of manifolds), and the analogous statement for $\text{Pos}_*^{\text{spin}}(pt)$.

The “shrinking” argument also works in the case of a “twisted product” [St1].

1.7 PROPOSITION. *Let g be a positive scalar curvature metric on a manifold M , and let $p: E \rightarrow N$ be a fiber bundle with fiber M whose structure group acts on M by isometries. Then E admits a positive scalar curvature metric.*

Now let’s consider positive scalar curvature metrics on simply connected manifolds. The relevant bordism group then is Ω_n^{spin} for spin manifolds and Ω_n^{SO} for non-spin manifolds. The oriented bordism ring Ω_*^{SO} was computed by Wall [Wa], who also constructed manifolds which represent multiplicative generators of this ring. The nice thing is that all these manifolds are projective bundles of (real or complex) vector bundles. In particular, they carry positive scalar curvature metrics by Proposition 1.7, and hence the Bordism Theorem implies the following result.

1.8 THEOREM (GROMOV-LAWSON [GL2, Cor. C]). *Let M be a simply connected manifold of dimension $n \geq 5$, which does not admit a spin structure. Then M carries a positive scalar curvature metric.*

For spin manifolds the story is more complicated, since not every spin manifold has a positive scalar curvature metric, as we’ll see in the next section.

2. Obstructions to positive scalar curvature metrics

2.1 THEOREM (LICHNEROWICZ [Li], 1963). *Let M be a spin manifold of dimension $n = 4k$, which has a positive scalar curvature metric g . Then the \hat{A} -genus $\hat{A}(M)$ vanishes.*

The \hat{A} -genus of an orientable manifold M is a rational number, obtained by evaluating a certain polynomial in the Pontrjagin classes of M on the fundamental class of M .

PROOF. If M is a spin manifold, the Atiyah-Singer Index Theorem implies

$$\hat{A}(M) = \text{index}(D) \stackrel{\text{def}}{=} \dim \ker D - \dim \text{coker } D,$$

where D is the “Dirac operator” on M . On the other hand, it follows from the “Weitzenböck formula” that the Dirac operator is invertible if the Riemannian metric used in the construction of D has positive scalar curvature, and in particular the index of D is zero in that case. \square

Later Hitchin found additional obstructions to the existence of positive scalar curvature metrics on spin manifolds of dimension $n \equiv 1, 2 \pmod{8}$ [Hi]. Hitchin uses a generalization of the Dirac operator which is called the “ $C\ell_n$ -linear Atiyah-Singer operator” in the book of Lawson-Michelsohn [LM]. It commutes with an action of the Clifford algebra $C\ell_n$ and has a “Clifford index” in $KO_n(pt)$ (cf. [LM], Ch. II, §7). We will use the notation $\alpha(M) \in KO_n(pt)$ for the (Clifford) index of the Atiyah-Singer operator on an n -dimensional spin manifold M . Making use again of the “Weitzenböck formula” one concludes:

2.2 THEOREM (HITCHIN [Hi]). *If M is a spin manifold which admits a positive scalar curvature, then $\alpha(M)$ vanishes.*

2.3 REMARK. To compare Theorems 2.1 and 2.2, we recall that by Bott periodicity the groups $KO_n(pt)$ are as follows.

$$KO_n(pt) = \begin{cases} \mathbb{Z} & n \equiv 0 \pmod{4} \\ \mathbb{Z}/2 & n \equiv 1, 2 \pmod{8} \\ 0 & \text{otherwise} \end{cases}$$

If n is divisible by 4, $\hat{A}(M)$ and $\alpha(M)$ agree up to a factor (cf. [LM], Ch. II, Thm. 7.10), and hence Hitchin’s result implies Lichnerowicz’s result. But Hitchin’s result is more general, since in dimensions $n \equiv 1, 2 \pmod{8}$, there are spin manifolds M with $\alpha(M) \neq 0$. In fact, every spin manifold of dimension $n \equiv 1, 2 \pmod{8}$, $n \geq 9$, is homeomorphic to one with non-trivial α -invariant [LM], Ch. IV, Cor. 4.2. This follows from the fact that the homotopy sphere Σ^n corresponding to an element in Adams’ μ -family has non-trivial α -invariant, and hence for any spin manifold M either M or the connected sum $M \# \Sigma$ (which is homeomorphic to M) has a non-trivial α -invariant. This shows that the question whether a manifold admits a positive scalar curvature metric is pretty subtle: the answer in general depends on the differentiable structure of M .

Hitchin’s result was generalized by Rosenberg who constructs a “Dirac operator” whose index is an element of the K -theory of a C^* -algebra. More precisely, let M be a spin manifold of dimension n , and let $f: M \rightarrow B\pi$ be a map to the classifying space of a discrete group π (not necessarily the fundamental group of M , but that is the main case of interest). Then Rosenberg constructs a “Dirac operator” D with $\text{index}(D)$ in $KO_n(C^*(\pi)) \stackrel{\text{def}}{=} \pi_n(BGL(C^*(\pi)))$. Here $C^*(\pi)$ is the (reduced) group C^* -algebra of π , which is a suitable completion of the real group ring $\mathbb{R}\pi$, and $BGL(C^*(\pi))$ is the classifying space of the general linear group of $C^*(\pi)$ (regarded as a topological group).

It turns out that $\text{index}(D) \in KO_n(C^*(\pi))$ is independent of the metric used in the construction of D . We will use the notation $\alpha(M, f)$ for $\text{index}(D)$, which agrees with $\alpha(M)$ if π is the trivial group. The Weitzenböck formula shows again that the index vanishes if the metric has positive scalar curvature, which proves:

2.4 THEOREM (ROSENBERG [Ro3]). *If M is a spin manifold of positive scalar curvature, and $f: M \rightarrow B\pi$ is a map to the classifying space of a discrete group π , then $\alpha(M, f)$ vanishes.*

3. The Gromov-Lawson-Rosenberg conjecture

3.1 CONJECTURE (GROMOV-LAWSON [GL3], ROSENBERG [Ro2]). *Let M be a spin manifold of dimension $n \geq 5$ with fundamental group π , and let $u: M \rightarrow B\pi$ be the classifying map of the universal covering $\widetilde{M} \rightarrow M$. Then M has a positive scalar curvature metric if and only if the element $\alpha(M, u)$ in $KO_n(C^*(\pi))$ vanishes.*

Using the Bordism Theorem 1.5 and the fact that $\alpha(M, f)$ depends only on the bordism class $[M, f]$ we can formulate the conjecture equivalently as follows:

3.2 CONJECTURE. $\text{Pos}_n^{\text{spin}}(B\pi)$ is the kernel of the homomorphism

$$\alpha: \Omega_n^{\text{spin}}(B\pi) \rightarrow KO_n(C^*(\pi)).$$

We note that $\text{Pos}_n^{\text{spin}}(B\pi)$ is contained in $\ker \alpha$ by Theorem 2.4. To prove the converse, one has to show that every class in $\ker \alpha$ can be represented by a manifold with positive scalar curvature. The problem is that even for $\pi = \{1\}$ we don’t have explicit manifolds which generate $\ker \alpha$ (or Ω_*^{spin} , for that matter, unlike the case of the oriented bordism ring). There are however partial results: the kernel of α is trivial in dimensions $n < 8$, and is the infinite cyclic group generated by the bordism class of the quaternionic projective space $\mathbb{H}\mathbb{P}^2$ for $n = 8$. It follows that the conjecture is true for $n \leq 8$. By finding explicit positive scalar curvature manifolds generating $\ker \alpha$ for $n < 23$, Rosenberg [Ro3] proved the conjecture in that range. Similarly, Miyazaki [Miy] found positive scalar curvature manifolds generating $\ker \alpha \otimes \mathbb{Z}[\frac{1}{2}]$, thus proving the conjecture after “inverting 2”.

Later Stolz gave the following characterization of the kernel of α .

3.3 THEOREM (STOLZ [St1]). *The kernel of $\alpha: \Omega_n^{spin} \rightarrow KO_n(pt)$ is equal to the subgroup T_n consisting of those bordism classes represented by total spaces of $\mathbb{H}\mathbb{P}^2$ -bundles, i.e. fiber bundles with fiber $\mathbb{H}\mathbb{P}^2$ and structure group the projective symplectic group $PSp(3)$.*

We note that the group $G = PSp(3)$ acts by isometries on $\mathbb{H}\mathbb{P}^2$ equipped with its standard metric. Hence these total spaces have positive scalar curvature metrics by Proposition 1.7, and thus the above result implies the conjecture in the simply connected case. In [St1], the equality $[\ker \alpha]_n = T_n$ is actually only proved localized at 2. This is enough for the proof of the conjecture by Miyazaki's result. Localized at odd primes, the above theorem was proved in [KrSt], using explicit manifold constructions ($\mathbb{H}\mathbb{P}^2$ -bundles over products of two quaternionic projective spaces).

SKETCH OF PROOF. The proof at the prime 2 is much more indirect and proceeds by first translating the problem into stable homotopy theory.

The group T_n can be described as the image of the homomorphism

$$\Psi: \Omega_{n-8}^{spin}(BG) \rightarrow \Omega_n^{spin},$$

which maps a bordism class $[N^{n-8}, f]$ to $[\widehat{N}]$, where $\widehat{N} \rightarrow N$ is the pull back of the 'universal' $\mathbb{H}\mathbb{P}^2$ -bundle $EG \times_G \mathbb{H}\mathbb{P}^2 \rightarrow BG$ via f .

Now consider the following commutative diagram.

$$(3.4) \quad \begin{array}{ccccc} \Omega_{n-8}^{spin}(BG) & \xrightarrow{\Psi} & \Omega_n^{spin} & \xrightarrow{\alpha} & KO_n(pt) \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ \pi_n(MSpin \wedge \Sigma^8 BG_+) & \xrightarrow{T_*} & \pi_n(MSpin) & \xrightarrow{D_*} & \pi_n(ko) \\ \widehat{T}_* \downarrow & & \parallel & & \parallel \\ \pi_n(\widehat{MSpin}) & \longrightarrow & \pi_n(MSpin) & \xrightarrow{D_*} & \pi_n(ko). \end{array}$$

Here the second row is the homotopy theoretic interpretation of the first row; the left and middle vertical isomorphism is given by the Pontrjagin-Thom construction. We recall that the Pontrjagin-Thom construction gives an isomorphism between the spin bordism group $\Omega_n^{spin}(X)$ of a space X and the n -th homotopy group of the spectrum $MSpin \wedge X_+$, where $MSpin$ is the Thom spectrum over $BSpin$, and X_+ is the union of X and a disjoint base point. For $n \geq 0$, $KO_n(pt)$ is isomorphic to the n -th homotopy group of a connective spectrum ko , called the *connective real K-theory spectrum*. Moreover, there are spectrum maps T (a kind of transfer map) and D (the ko -orientation of $MSpin$), whose induced maps in homotopy make the upper two squares commutative.

Using the families index theorem, it is proved that the composition $D \circ T$ is homotopic to the constant map, and hence T factors through a map \widehat{T} from $MSpin \wedge \Sigma^8 BG_+$ to \widehat{MSpin} , the homotopy fiber of D . The bottom row is part of the long exact homotopy sequence of this fibration.

The statement of the theorem is equivalent to saying that the top row of the diagram is exact. By exactness of the bottom row it suffices to show that \widehat{T}_* is surjective at the prime 2. The proof of this uses the following facts.

- (1) The homomorphism induced by \widehat{T} on $\mathbb{Z}/2$ -cohomology is a split injection of modules over the Steenrod algebra.
- (2) The Adams spectral sequence converging to the 2-local homotopy groups of $MSpin \wedge BG_+$ collapses.

By (1) the map of Adams spectral sequences induced by \widehat{T} is a surjection on the E_2 -level, by (2) on the E_∞ -level, and hence \widehat{T} is surjective on the 2-local homotopy groups. \square

Now we turn to discuss Conjecture 3.1 in the case of a non-trivial fundamental group. It might be tempting to think that a manifold M has a positive scalar curvature metric if and only if its universal covering does, at least if $\pi_1(M)$ is finite, by arguing one should be able to “average” a positive scalar curvature metric on the universal covering of M to get a $\pi_1(M)$ -equivariant positive scalar curvature metric which then descends to a positive scalar curvature metric on M . However, averaging a positive scalar curvature metric might not give a positive scalar curvature metric as the example below shows. On the other hand, if $\pi_1(M)$ is finite of *odd order*, then it can be shown that the vanishing of $\alpha(\widetilde{M})$ implies the vanishing of $\alpha(M, u)$, and hence Conjecture 3.1 claims in this case that M has a positive scalar curvature metric if and only if \widetilde{M} does!

3.5 EXAMPLE (BERGERY-BERARD [BB], EXAMPLE 9.1). Let Σ be a 9-dimensional homotopy sphere with $\alpha(\Sigma) \neq 0$ (cf. Remark 2.3), and let M be the connected sum $(\mathbb{R}P^7 \times S^2) \# \Sigma$. We note that the real projective space $\mathbb{R}P^7$ and hence $\mathbb{R}P^7 \times S^2$ are spin manifolds. It follows that the α -invariant of $\mathbb{R}P^7 \times S^2$ vanishes, since it has a positive scalar curvature metric. Noting that α is additive with respect to connected sum (α is a bordism invariant, and the connected sum is bordant to the disjoint union), we get:

$$\alpha(M) = \alpha(\mathbb{R}P^7 \times S^2) + \alpha(\Sigma) \neq 0.$$

Hence by Theorem 2.2 M does not admit a positive scalar curvature metric. On the other hand, we have

$$\widetilde{M} \cong S^7 \times S^2 \# \Sigma \# \Sigma \cong S^7 \times S^2,$$

since the group of 9-dimensional homotopy spheres is isomorphic to $(\mathbb{Z}/2)^3$, and hence $\Sigma \# \Sigma$ is diffeomorphic to S^7 . So we see that \widetilde{M} has a positive scalar curvature metric.

Attempting to prove Conjecture 3.1/3.2, an understanding of $\ker \alpha$ is crucial. Hence the following factorization of α is useful.

$$(3.6) \quad \Omega_n^{spin}(B\pi) \xrightarrow{D_*} ko_n(B\pi) \xrightarrow{p} KO_n(B\pi) \xrightarrow{A} KO_n(C^*(\pi))$$

Here p is the canonical map from connective to periodic KO -homology, and A is the “assembly map”.

3.7 THEOREM (JUNG [Ju], STOLZ [St2]). *Let M be a spin manifold of dimension $n \geq 5$ with fundamental group π . Let $u: M \rightarrow B\pi$ be the classifying map of the universal covering $\widetilde{M} \rightarrow M$. Then M has a positive scalar curvature metric if and only if $D_*([M, u]) \in \text{Pos}_n^{ko}(B\pi)$, where $\text{Pos}_n^{ko}(B\pi)$ is the image of D_* restricted to $\text{Pos}_n^{spin}(B\pi)$.*

SKETCH OF PROOF. It suffices to show $\ker D_* \subseteq \text{Pos}_n^{spin}(B\pi)$. Away from the prime 2 this is proved by Jung, who gives a Baas-Sullivan description of $ko_*(X) \otimes \mathbb{Z}[\frac{1}{2}]$. In particular, $[M, u] \in \ker D_*$ implies that the connected sum of 2^r copies of (M, u) for some r bounds a manifold with singularities, and Jung uses this to construct a positive scalar curvature metric on the connected sum.

The result at the prime 2 is due to Stolz, who proves that an odd multiple of a bordism class in the kernel of D_* can be represented by the total space of an $\mathbb{H}\mathbb{P}^2$ -bundle. This boils down to the homotopy theoretic statement that the middle row in diagram 3.4 is still exact after smashing with $B\pi_+$. This follows from the fact that the map \widehat{T} is a split surjection of spectra, which in turn is proved by splitting the spectra $M\text{Spin} \wedge \Sigma^8 BG_+$ and $\widehat{M\text{Spin}}$ using Adams spectral sequence techniques. \square

The above result is a significant improvement compared to the Bordism Theorem 1.5, since the connective KO -theory groups $ko_*(B\pi)$ are *much* smaller than $\Omega_*^{spin}(B\pi)$. Unfortunately, $ko_*(B\pi)$ has been computed for only a handful of finite groups, notably cyclic groups [Ha], elementary abelian 2-groups [Yu], and the quaternion and dihedral group of order 8 [Ba]. Still, we immediately obtain the following.

3.8 COROLLARY. *The conjecture 3.1 is true if p and A are injective.*

Note that many torsion-free groups π for which A is injective were listed in [Ro3]. (This is related to the Novikov conjecture, as we will note below.) If in addition $B\pi$ is stably a wedge of spheres, then p is clearly a split injection. Thus Corollary 3.8 applies to free groups, free abelian groups, fundamental groups of orientable surfaces, and many similar examples.

3.9 THEOREM (ROSENBERG-STOLZ [RS1], THM. 5.3(4)). *The conjecture 3.1 is true for $\pi \cong \mathbb{Z}/2$.*

SKETCH OF PROOF. An Adams spectral sequence calculation shows that the kernel of $A \circ p: ko_n(B\pi) \rightarrow KO_n(C^*(\pi))$ is trivial for $n \not\equiv 3 \pmod{4}$, and a finite cyclic group generated by $D_*[\mathbb{R}\mathbb{P}^n, u]$ for $n \equiv 3 \pmod{4}$. This implies the conjecture by Theorem 3.7. \square

4. The stable conjecture

The real K -theory groups of a real C^* -algebra A are 8-periodic. Moreover, the isomorphism $KO_n(A) \xrightarrow{\cong} KO_{n+8}(A)$ is given by multiplication by the generator b of $KO_8(pt) = KO_8(\mathbb{R}) \cong \mathbb{Z}$. We can find a simply connected spin manifold B of dimension 8 with $\alpha(B) = b$. There are many possible choices for B , but we just

pick one, and call it the “Bott manifold”. The Bott periodicity for $KO_*(C^*(\pi))$ shows that given a manifold M with fundamental group π , $\alpha(M, u)$ vanishes if and only if $\alpha(M \times B, u)$ vanishes (we use the letter u for the classifying map of the universal covering of whatever manifold we are talking about). This shows that Conjecture 3.1 is equivalent to the following two conjectures:

4.1 CANCELLATION CONJECTURE. *Let M be a spin manifold of dimension $n \geq 5$. Then M has a positive scalar curvature metric if and only if $M \times B$ does.*

4.2 STABLE CONJECTURE. *Let M be a spin manifold. Then M has stably a positive scalar curvature metric (i.e. the product of M with sufficiently many copies of B has a positive scalar curvature metric) if and only if $\alpha(M, u) = 0$.*

An important tool for proving the Stable Conjecture is the following geometric description of periodic KO -homology.

4.3 THEOREM (KRECK-STOLZ [KrSt], THM. C). *Given a space X , let $T_*(X)$ be the subgroup of $\Omega_*^{spin}(X)$ represented by pairs $(\widehat{N}, f \circ p)$, where $\widehat{N} \xrightarrow{p} N$ is an $\mathbb{H}\mathbb{P}^2$ -bundle, and $f: N \rightarrow X$ is a map. Let $b \in \Omega_8^{spin}(pt)/T_8(pt) \cong ko_8(pt) \cong \mathbb{Z}$ be the generator, i.e., the class represented by the Bott manifold. Then the homomorphism $p \circ D_*: \Omega_*^{spin}(X) \rightarrow KO_*(X)$ induces an isomorphism between the groups $\Omega_*^{spin}(X)/T_*(X)[b^{-1}]$ and $KO_*(X)$.*

This shows in particular that if $[M, u] \in \Omega_*^{spin}(X)$ is in the kernel of $p \circ D_*$, then the product $M \times B \times \dots \times B$ with sufficiently many copies of B represents an element in $T_n(X)$, and hence carries a positive scalar curvature metric. This implies the following result.

4.4 COROLLARY. *If the assembly map $A: KO_n(B\pi) \rightarrow KO_n(C^*(\pi))$ is injective, then the Stable Conjecture 4.2 is true.*

The Novikov conjecture (or rather, a form of it) claims that A is injective for torsion free groups. It has been proved for many groups, notably for torsion free, discrete subgroups of Lie groups [Ka]. The assembly map is definitely not injective for some groups, e.g., finite groups. In general, we obtain the following consequence of Theorem 4.3.

4.5 COROLLARY. *Let M be a spin manifold of dimension $n \geq 5$ with fundamental group π . Let $u: M \rightarrow B\pi$ be the classifying map of the universal covering $\widetilde{M} \rightarrow M$. Then M has stably a positive scalar curvature metric if and only if $p \circ D_*([M, u])$ is in $\text{Pos}_n^{KO}(B\pi)$, where $\text{Pos}_n^{KO}(B\pi) \subset KO_n(B\pi)$ is the image of $\bigoplus_{k \geq 0} \text{Pos}_{n+8k}^{spin}(B\pi)$ under $p \circ D_*$ (here we identify $KO_{n+8k}(X)$ with $KO_n(X)$ using periodicity).*

We would like to point out the formal similarities between this result, the Bordism Theorem 1.5, and Theorem 3.7. However, $KO_*(B\pi)$ is much easier to compute than $ko_*(B\pi)$ or $\Omega_*^{spin}(B\pi)$. For example, $KO_*(B\pi)$ for a finite group π can be expressed in terms of the representation ring of π , in a fashion similar to

Atiyah-Segal's calculation of $KO^*(B\pi)$ [AS]. In fact, we obtain the description of $KO_*(B\pi)$ by dualizing their result.

4.6 THEOREM (ROSENBERG-STOLZ [RS2]). *The Stable Conjecture 4.2 is true for finite groups π .*

SKETCH OF PROOF. By Corollary 4.5 it suffices to show $\ker A \subseteq \text{Pos}_*^{KO}(B\pi)$. Using a result of Kwasik-Schultz [KwSc], (cf. [RS1], Prop. 5.2) we can assume that π is a p -group. Also it is enough to work with $\widetilde{KO}_*(B\pi)$, since $\ker A = \text{Pos}_*^{KO}(B\pi)$ is true if π is the trivial group. As mentioned before, the groups $\widetilde{KO}_*(B\pi)$ for a p -group π can be calculated. The result says in particular that $\widetilde{KO}_*(B\pi)$ is a direct sum of finitely many copies of \mathbb{Z}/p^∞ plus, for $p = 2$, finitely many copies of $\mathbb{Z}/2$. Moreover, the kernel of the assembly map restricted to $\widetilde{KO}_*(B\pi)$ consists precisely of the \mathbb{Z}/p^∞ 's, and we have to show that all those elements can be represented by positive scalar curvature manifolds.

This is proved first for cyclic groups π of order $k = p^r$. We note that the classifying space $B\mathbb{Z}/k$ can be identified with the sphere bundle $S(H^{\otimes k})$ of the k -th tensor power of the Hopf line bundle over complex projective space $\mathbb{C}\mathbb{P}^\infty$. Then the homotopy cofibration

$$B\mathbb{Z}/k = S(H^{\otimes k}) \xrightarrow{q} \mathbb{C}\mathbb{P}^\infty \rightarrow T(H^{\otimes k}),$$

where $T(H^{\otimes k})$ denotes the Thom space of $H^{\otimes k}$, induces a long exact sequence of KO -homology groups

$$\rightarrow KO_{n+1}(T(H^{\otimes k})) \xrightarrow{\partial} \widetilde{KO}_n(B\mathbb{Z}/k) \xrightarrow{q_*} \widetilde{KO}_n(\mathbb{C}\mathbb{P}^\infty) \rightarrow .$$

The group $\widetilde{KO}_n(B\mathbb{Z}/k)$ is a torsion group, while $\widetilde{KO}_n(\mathbb{C}\mathbb{P}^\infty)$ is torsion free. Hence q_* is trivial, and ∂ is surjective.

Using again the geometric description of KO_* -homology in Theorem 4.3, one can show that the image of ∂ can be represented by total spaces of S^1 -bundles over simply connected manifolds. Moreover, these manifolds are non-spin if p is odd. Hence in this case these manifolds and consequently the total spaces of the S^1 -bundles over them admit positive scalar curvature metrics, which shows $\widetilde{KO}_n(B\mathbb{Z}/k) = \widetilde{\text{Pos}}_n^{KO}(B\mathbb{Z}/k)$.

For $p = 2$, $H^{\otimes k}$ is a spin bundle and hence using the Thom isomorphism we get

$$KO_{n+1}(T(H^{\otimes k})) \cong KO_{n-1}(\mathbb{C}\mathbb{P}^\infty) \cong \widetilde{KO}_{n-1}(\mathbb{C}\mathbb{P}^\infty) \oplus KO_{n-1}(pt).$$

The image of ∂ restricted to $KO_{n-1}(pt)$ is a finite subgroup, and hence ∂ restricted to $\widetilde{KO}_{n-1}(\mathbb{C}\mathbb{P}^\infty)$ is still surjective on the \mathbb{Z}/p^∞ summands. But the image of ∂ restricted to $\widetilde{KO}_{n-1}(\mathbb{C}\mathbb{P}^\infty)$ is again represented by total spaces of S^1 -bundles over manifolds with positive scalar curvature. This proves the theorem in the case of cyclic groups.

For the general case we use the fact that for a finite group π the representations induced up from cyclic subgroups span a subgroup of finite index of

the representation ring of π . It follows that the image of $\bigoplus_H KO_*(BH)$ in $KO_*(B\pi)$, where H runs through all cyclic subgroups of π , has finite index. In particular the kernel of α is in the image, which reduces the general case to the cyclic case. \square

So far, we only discussed what is known about *spin* manifolds with positive scalar curvature and the group $\text{Pos}_n^{\text{spin}}(B\pi)$, but not the analogous group $\text{Pos}_n^{\text{SO}}(B\pi)$, which according to the Bordism Theorem 1.5 determines whether an orientable manifold, whose universal cover is non-spin, has a positive scalar curvature metric. We observe that for such a manifold M the product $M \times B$ always has a positive scalar curvature metric, since B represents the same bordism class in $\Omega_8^{\text{SO}}(pt)$ as the disjoint union of 64 copies of $\mathbb{C}\mathbb{P}^2 \times \mathbb{C}\mathbb{P}^2$. This implies

$$[M \times B] = [M \times \coprod_{64} \mathbb{C}\mathbb{P}^2 \times \mathbb{C}\mathbb{P}^2] \in \text{Pos}_{n+8}^{\text{SO}}(B\pi),$$

since $\coprod_{64} \mathbb{C}\mathbb{P}^2 \times \mathbb{C}\mathbb{P}^2$ and hence $M \times \coprod_{64} \mathbb{C}\mathbb{P}^2 \times \mathbb{C}\mathbb{P}^2$ has a positive scalar curvature metric. Then the Bordism Theorem shows that $M \times B$ has a positive scalar curvature metric.

This shows in particular, that methods using suitable Dirac operators and their indices in some KO -groups don't work here to give obstructions for positive scalar curvature metrics on M . One might object here that M is not spin, and hence there is no Dirac operator on M anyway. However, one can define a Dirac operator [St3], which generalizes Rosenberg's Dirac operator and whose index lives in the KO -theory of a “twisted” version of $C^*(\pi_1(M))$, where the “twist” is determined by the first two Stiefel-Whitney classes of M . The manifold can be non-spin, or even non-orientable; the only condition needed for the construction of this operator is that the universal covering \widetilde{M} is spin.

At this point, one might believe that every orientable manifold M of dimension $n \geq 5$, whose universal cover is non-spin, has in fact a positive scalar curvature metric, since this is true in the simply connected case by Theorem 1.8, and there are no obstructions coming from the indices of Dirac operators. However, there is another technique for producing obstructions to positive scalar curvature metrics, namely the minimal hypersurface method of Schoen-Yau [SY]. Using it one can show that the connected sum $M = T^6 \# (\mathbb{C}\mathbb{P}^2 \times S^2)$ of the 6-dimensional torus with $\mathbb{C}\mathbb{P}^2 \times S^2$ does not admit a positive scalar curvature metric [GL3, p. 186], despite that fact that the universal covering \widetilde{M} is non-spin. At this point, the characterization of the subgroup $\text{Pos}_n^{\text{SO}}(B\pi) \subseteq \Omega_n^{\text{SO}}(B\pi)$ is wide open — to the authors' best knowledge there is not even a conjecture about it.

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