PRELIMINARY VERSION

Five Lectures on Supersymmetry

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February 4, 1998

This is an expanded writeup of five lectures given initially at the Institute for Theoretical Physics in Santa Barbara January 12–16 and then repeated at the Institute for Advanced Study in Princeton shortly thereafter. They are intended for mathematicians interested in learning some basics about supersymmetry. In addition to supersymmetry we discuss some basic notions of classical (and semiclassical) field theory. However, our entire discussion is kinematical in that it does not touch upon the dynamical behavior of the theories. Also, we have very little if anything to say about supergravity.

The lecture titles are:

Lecture 1: What Are Fermions?
Lecutre 2: Symmetry and Supersymmetry
Lecture 3: Supersymmetry in Various Dimensions
Lecture 4: Theories with Two Supersymmetries
Lecture 5: Theories with More Supersymmetries

Most, but not all, of the material we discuss has been known in the physics literature for 20 years or more. My understanding grew out of the program last year at the IAS. Lectures by Joseph Bernstein, Ed Witten, and Nati Seiberg and many discussions with them as well as with Pierre Deligne, John Morgan, and others shaped the material presented here.

In this preliminary version we do not supply references but will give some in the final version. More details will appear in the materials being prepared from last year's program, some of which are available on the Web at `http://www.math.ias.edu/~drm/QFT`.

As this is a preliminary version, please send comments, corrections, complaints, etc. to me at `dafr@math.utexas.edu`.

The author is supported by . . . .

Lecture 1: What Are Fermions?

Before we can talk about supersymmetry, which is a symmetry between bosons and fermions, we need to come to grips with the concept of a fermion. We begin in the quantum world by describing a particle as an irreducible representation of an appropriate symmetry group. Particles are classified as either bosons or fermions, and the distinction is neatly encoded by the mathematics of $\mathbb{Z}/2$-graded algebra. It is customary to use 'super' in place of '$\mathbb{Z}/2$-graded' and so refer to this as SuperAlgebra. We review the basic idea—the sign rule—and show how auxiliary odd parameters may be introduced as a convenient computational device. The mathematics of quantum fermions is completely standard: a very mildly noncommutative form of algebra.

In the second half of the lecture we recall how to pass from a classical bosonic particle to a quantum Hilbert space. (We will deal with fields in subsequent lectures.) The common saying is "Quantization is an art, not a functor" and in these lectures we have nothing to add to the vast literature on this subject. In fact, we will always quantize affine spaces where there is a good mathematical theory. After reviewing the standard bosonic particle, whose quantization gives a quantum boson, we try to construct a classical model whose quantization also involves quantum fermions. Thus we are led to what we might call "classical fermions." These are the infamous anticommuting variables which give differential geometers the fits. The main point is to understand that if

$$\psi\colon \mathbb{R} \longrightarrow \mathbb{R}$$

is a compactly supported map, then the expression

$$\int_{\mathbb{R}} \psi(t)\dot{\psi}(t)\, dt$$

can be nonzero. At first glance this looks impossible, since the integrand appears to be the exact differential $d(\psi(t)^2/2)$. A physicists' response is, "ψ is anticommuting your integration by parts has a sign error!" By the end of the lecture we will understand this response in mathematical terms and be able to nod agreeably.

The reader who is happy to work with anticommuting functions can skip this part of the lecture, but for the rest of us it is a bit of psychotherapy: After the mathematical explanation we can compute with these anticommuting variables and not worry about it!

Fermions in Quantum Mechanics

The state of a quantum mechanical system is represented by a vector in a *complex* Hilbert space. More concretely, it is a wave function and in a first course in quantum mechanics we learn that the complex values allow for the interference characteristic of quantum phenomena. Usually the wave function is normalized to have unit norm, and wave functions which differ by a phase represent the same physical state. We study both nonrelativistic and relativistic quantum mechanics.

In the nonrelativistic case a single particle in $\mathbb{R}^m$ is modeled by the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}^m, W),$$

where W is an irreducible complex representation of the rotation group $\mathrm{Spin}(m)$. (Recall that $\mathrm{Spin}(m)$ is the double cover of the rotation group $SO(m)$.) In the simplest case $m = 1$ and W is the trivial representation. Then $\mathcal{H}$ is the space of complex-valued L^2 functions on the line. It is the Hilbert space which underlies the free particle on the line or the harmonic oscillator, the first examples one usually encounters. In 3 space dimensions ($m = 3$) we might also take W to be the (complexification of the) standard 3 dimensional representation of the rotation group $SO(3)$. The corresponding particle is a massive spin 1 particle. If we take instead the 2 dimensional spin representation of $\mathrm{Spin}(3) \cong SU(2)$ then we obtain a Hilbert space of the massive spin 1/2 particle.

What do we mean by the "spin" of a representation of $\mathrm{Spin}(m)$? Suppose W is such a representation. Fix a 2-plane in $\mathbb{R}^m$ and consider the subgroup $SO(2) \subset SO(m)$ of rotations in that plane which fix the perpendicular plane. We work with the double cover $\mathrm{Spin}(2) \subset \mathrm{Spin}(m)$. Under $\mathrm{Spin}(2)$ the representation W decomposes as a sum of one dimensional complex representation on each of which $\mathrm{Spin}(2)$ acts by $\lambda \mapsto \lambda^{2j}$, where $\lambda \in \mathrm{Spin}(2)$ and j is a half-integer. (We use half-integers so that $j = 1$ is the identity representation of the $SO(2)$ subgroup of $SO(m)$.) The spin of the representation W is the largest j which occurs in the decomposition.

For example, the trivial representation has spin 0. It represents a *scalar particle*. The m dimensional defining representation of $SO(m)$ has spin 1; the corresponding particle is sometimes called the *vector particle*. The reader can check that all exterior powers of this representation also have spin 1. The spin representation of $\mathrm{Spin}(m)$ has spin 1/2. One obtains higher spin by looking at the symmetric powers of the defining representation.

There are physical reasons why in interacting local quantum field theories one only sees particles of low spin. More precisely, in theories without gravity only particles of spin 0, spin 1/2, and spin 1 occur. Spin 1 particles only occur in *gauge theories*; a theory with only spin 0 and spin 1/2 particles is a *σ-model*. The *graviton*—the particle which mediates the gravitational force—is a particle of spin 2 and in theories of supergravity there are also particles of spin 3/2. That's it! There are no particles of higher spin in realistic theories.

In nonrelativistic quantum mechanics one also has on the Hilbert space a representation of translations in space and a Hamiltonian. Together the data generates a representation of (a central extension of) the Galilean group, but we will not need this. In the *relativistic* case the Galilean group is replaced by the Poincaré group, and we will generalize in Lecture 3 when we study supersymmetries in general. So we take this opportunity to review some basics about representations of the Poincaré group.

Let $V = \mathbb{R}^{1,n-1}$ be a vector space with coordinates $t, x^1, x^2, \dots, x^{n-1}$ and Lorentz metric

$$g = c^2(dt)^2 - (dx^1)^2 - \cdots - (dx^{n-1})^2. \tag{1.1}$$

Here c is the speed of light, t represents time, and the x^i are the spatial coordinates. We often set

$$x^0 = ct$$

for convenience. In the nonrelativistic case $\mathbb{R}^m$ represents space, so we should think that the spacetime dimension n is equal to $m+1$. *Minkowski space* M^n is the affine space which underlies V. It has a constant metric given by (1.1). At each point we have the *lightcone* of *lightlike* vectors of zero norm. Vectors inside the lightcone are *timelike* and have positive norm; vectors outside the lightcone are *spacelike* and have negative norm. (See Figure 1.)

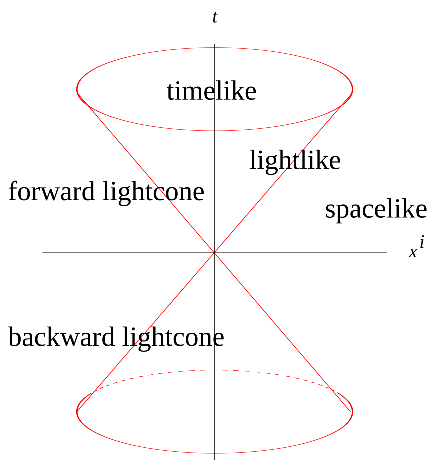

Figure 1: The Lorentzian vector space V

The group of isometries of V is the orthogonal group $O(1, n-1)$. It has four components distinguished by the determinant (± 1) and whether or not the forward lightcone is preserved or is mapped to the backward lightcone. The double cover of the identity component is the spin group $\mathrm{Spin}(1, n-1)$. The group of isometries of Minkowski space M^n includes the subgroup of translations V and the quotient by this subgroup is isomorphic to $O(1, n-1)$. Perhaps that group of isometries is usually called the Poincaré group. Be that as it may, we call the Poincaré group P^n the double cover of the identity component of that group. It fits into the exact sequence

$$1 \to V \to P^n \to \mathrm{Spin}(1, n-1) \to 1.$$

Now a single relativistic quantum particle is defined to be an irreducible unitary representation $\mathcal{H}$ of P^n. Such representations were classified by Wigner long ago, and we quickly review the construction. First, we restrict the representation to the translation subgroup V. Since V is abelian the representation decomposes as a direct sum (really direct integral) of one dimensional representations on which V acts by a character

$$v \longmapsto \text{multiplication by } e^{i\alpha(v)}, \qquad \alpha \in V^*.$$

The set of infinitesimal characters α which occur are permuted by the action of $SO(1, n-1)$ on V^*. Since the representation $\mathcal{H}$ is irreducible these infinitesimal characters form an orbit of the action. There are three types of orbits which have nonnegative energy. (Nonnegative energy is a physical requirement.) These are indicated in Figure 2. Notice that the axes are labeled E for energy and p_i for momentum. Energy and momentum are the dual variables to time and space. Also, we have the relativistic mass formula

$$\frac{E^2}{c^4} - \sum_i \frac{p_i^2}{c^2} = m^2$$

for the mass square. The mass is constant on an orbit and so is an invariant of an irreducible representation. The simplest orbit is a single point, the origin. It corresponds to the trivial representation $\mathcal{H} = \mathbb{C}$ which represents the *vacuum* in a theory. It is not a representation of a single particle. The other two orbits correspond to *massless* $(m = 0)$ and *massive* $(m > 0)$ representations.

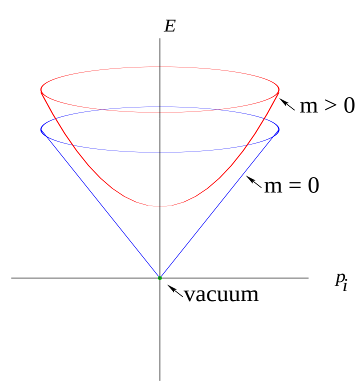

FIGURE 2: ORBITS OF $SO(1, n-1)$ IN V^*, $n \geq 3$

Warning: In the two dimensional case $(n = 2)$ the massless orbit breaks up into two distinct orbits along the two rays of the positive lightcone, as indicated in Figure 3. We term call them

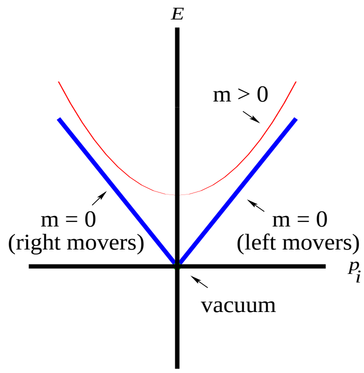

Figure 3: Orbits of $SO(1,1)$ in V^*

right movers and *left movers* since the corresponding characters are functions of $ct - x$ and $ct + x$ respectively. So there is a richer classification of massless particles in two dimensions.

Now the representation $\mathcal{H}$ of P^n is obtained by constructing a homogeneous complex hermitian vector bundle over the orbit. More precisely, the total space of the bundle carries an action of $\mathrm{Spin}(1, n-1)$ covering the action of $SO(1, n-1)$ on the orbit. Such bundles may be constructed by fixing a point on the orbit and constructing a finite dimensional unitary representation of the stabilizer subgroup of that point, called in this context the *little group*.

Consider first the massive case and fix the mass to be m. For convenience we set $c = 1$ and take as basepoint $(m, 0, \ldots, 0)$. Then the stabilizer subgroup, or little group, is easily seen to be $\mathrm{Spin}(n-1)$. Thus a massive particle corresponds to a representation of $\mathrm{Spin}(n-1)$, just as in the nonrelativistic case.

In the massless case we consider the basepoint $(1, 1, 0, \ldots, 0)$. The stabilizer group in this case is a double cover of the Euclidean group of orientation preserving isometries of an $n-2$ dimensional Euclidean space. We can see this as follows. The group $SO(1, n-1)$ is the group of conformal transformations of the $n-2$ dimensional sphere. We can view the sphere as the intersection of the forward lightcone with a hyperplane where E is constant. An isometry preserves these hyperplanes; a dilation moves them according to the dilation factor. In this way these conformal transformations act on the forward lightcone. Now consider the basepoint to be “infinity” in the sphere at $E = 1$. Then the conformal transformations which preserve that point are isometries which fix the point at infinity, i.e., the Euclidean group. Massless particles correspond to finite dimensional unitary representations of this group, and such representations are necessarily trivial on the subgroup of translations. In other words, they factor through to representations of $\mathrm{Spin}(n-2)$. Because of this we will usually refer to $\mathrm{Spin}(n-2)$ as the massless little group, even though this is only the reductive part of the stabilizer.

We have the notion of spin of a representation as discussed earlier.

Given this homogeneous vector bundle we take $\mathcal{H}$ to be the space of L^2 sections of the bundle over the orbit. (There is an $SO(1, n-1)$-invariant measure on the orbit.)

In either the massive or massless case the trivial representation corresponds to a spin 0 particle. A spin 1 particle corresponds to a representation of dimension $n-1$ in the massive case and $n-2$ in the massless case. Spin 1/2 particles correspond to spin representations of the little group; we discuss these more carefully in Lecture 3.

Remark: In a local quantum field theory the CPT theorem states that the representation of P^n extends to a (projective) representation of the larger group whose quotient by translations is the entire group $SO(1, n-1)$. (Recall that in P^n we only consider the identity component of transformations which preserve the forward light cone.) Elements in the new component are represented by antilinear maps. The condition that the representation extend can be stated in terms of the little group; the precise statement depends on the parity of n. For n even it says that the representation of the little group is self-conjugate, i.e., either real or quaternionic. The statement for n odd is more complicated and we omit it.

What we have described so far is the Hilbert space of states of a single particle. Suppose now we want to describe a two particle state. If the individual particles are represented by Hilbert spaces $\mathcal{H}, \mathcal{H}'$, then the two particle states are vectors in $\mathcal{H} \otimes \mathcal{H}'$. Now suppose that the two particles are of the same type: $\mathcal{H} = \mathcal{H}'$. Then if v, v' are vectors in $\mathcal{H}$ representing one particle states, the states $v \otimes v'$ and $v' \otimes v$ are identical. This is what we mean when we say that these are identical particles. But recall that identical states are represented by unit vectors which differ by a phase. So we have

$$v' \otimes v = \lambda\, v \otimes v' \tag{1.2}$$

for some phase λ. In other words, transposition of the factors leads to multiplication by λ. If we transpose twice we may suppose that we get back the original vector. In fact, there are more exotic possibilities in two dimensions, but we ignore them. Thus we will conly consider $\lambda = 1$ or $\lambda = -1$. A particle is a *boson* if $\lambda = 1$ and a *fermion* if $\lambda = -1$.

To summarize: If $\mathcal{H}_B$ is the Hilbert space of states of a single boson B, then the symmetric square $\operatorname{Sym}^2 \mathcal{H}_B$ is the space of states of two B particles. If $\mathcal{H}_F$ is the space of states of a single fermion F, then the exterior square $\bigwedge^2 \mathcal{H}_F$ is the space of states of two F particles.

There is a nice mathematical device to keep track of the difference between bosons and fermions. We discuss it in the next section. First a few remarks:

- The fact that a two fermion state lives in the exterior square encodes the *Pauli exclusion principle*, which states that two identical fermions cannot be in the same state. This is the reason why we cannot combine two microscopic fermions into a macroscopic object—we cannot put many of them in the same state. Bosons can combine this way, and so microscopic bosons combine to form the macroscopic objects we see in ordinary life. We might say that "there are no classical fermions."

- In unitary local quantum field theories there is a connection between the spin of a particle and its *statistics*—whether it is a boson or a fermion. A particle of integral spin is a boson and a particle of half-integral spin is a fermion. This spin-statistics connection is often violated in nonunitary theories, and in particular in the topological field theories which have mathematical applications. In our discussions of field theory (starting in Lecture 3) we will always work in Minkowski space and assume this spin-statistics connection. But it is violated in the superparticle example of Lecture 2.

SuperAlgebra

The idea of SuperAlgebra, or $\mathbb{Z}/2$-graded Algebra, is simple: Replace vector spaces by $\mathbb{Z}/2$-graded vector spaces and invoke the *sign rule* when commuting homogeneous elements. We assume that all structures (multiplications, etc.) are even.

Unless otherwise stated we use 'graded' for '$\mathbb{Z}/2$-graded'.

A *graded vector space*

$$W = W^0 \oplus W^1$$

is simply a vector space which is presented as the direct sum of two subspaces. Elements in W^0 are termed *even*, and elements in W^1 are termed *odd*. We call the evenness or oddness of a homogeneous element its *parity* and use '$|a|$' to denote the parity of the homogeneous element a. Thus

$$|a| = \begin{cases} 0, & \text{if } a \in W^0; \\ 1, & \text{if } a \in W^1. \end{cases}$$

The *sign rule* states that in any equation where we permute the homogeneous elements a and b we pick up a sign $(-1)^{|a||b|}$.

We illustrate with two representative cases.

Example 1 (commutative superalgebra). An ordinary algebra A is a vector space with a multiplication $A \otimes A \to A$ which satisfies the usual associative and distributive laws. There is also an identity element. We say A is commutative if

$$ab = ba \tag{1.3}$$

for all elements $a, b \in A$.

To derive from this the definition of a *commutative superalgebra* we need to make the underlying vector space graded, postulate that the multiplication and identity are even, and use the sign rule. In detail: A is now graded

$$A = A^0 \oplus A^1.$$

The evenness of the identity element means $1 \in A^0$. The evenness of the multiplication is the statement

$$A^0 \cdot A^0 \subset A^0, \qquad A^0 \cdot A^1 \subset A^1, \qquad A^1 \cdot A^1 \subset A^0. \tag{1.4}$$

Finally the sign rule applied to the commutative law (1.3) is the rule

$$ab = (-1)^{|a|\,|b|}\, ba \tag{1.5}$$

for homogeneous elements a, b. These are the axioms of a commutative superalgebra.

A common example of a commutative superalgebra is the exterior algebra of a vector space. It is actually $\mathbb{Z}$-graded, but we only remember the coarser $\mathbb{Z}/2$ grading in our current context. Equation (1.5) is the usual rule for the wedge product.

We will often consider the commutative superalgebra over $\mathbb{R}$ generated by anticommuting elements $\eta_1, \dots, \eta_N$. We denote it as

$$R = \mathbb{R}[\eta_1, \dots, \eta_N]. \tag{1.6}$$

The generators η_i satisfy

$$\eta_i \eta_j = -\eta_j \eta_i. \tag{1.7}$$

In particular, the square of any η is zero.

Example 2 (super Lie algebra). An ordinary Lie algebra is a vector space $\mathfrak{g}$ with a bracket $[\cdot,\cdot]\colon \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ which is skew-symmetric and satisfies the Jacobi identity. Following our prescription we define a *super Lie algebra* to be a graded vector space

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$$

together with a bracket. The bracket is even, which means it satisfies conditions analogous to (1.4). The skew symmetry of the bracket is now

$$[a, b] = (-1)^{|a|\,|b|}[b, a] \tag{1.8}$$

for homogeneous elements a, b. We leave the correct modification of the Jacobi identity to the reader—it has some ugly signs when written for three arbitrary homogeneous elements.

These axioms are equivalent to the following: $\mathfrak{g}^0$ is an ordinary Lie algebra, $\mathfrak{g}^1$ is a representation of $\mathfrak{g}^0$, and there is a $\mathfrak{g}^0$-invariant *symmetric* pairing $\mathfrak{g}^1 \otimes \mathfrak{g}^1 \to \mathfrak{g}^0$.

There is a super Lie algebra which one meets in differential geometry. Suppose ξ is a vector field on a manifold. Let $\mathrm{Lie}(\xi)$ denote the Lie derivative along ξ and $\iota(\xi)$ the contraction with ξ, both viewed as operators on differential forms. Then together with the exterior derivative d they generate a three dimensional graded Lie algebra over $\mathbb{R}$ with the only nontrivial bracket being the Cartan formula

$$\mathrm{Lie}(\xi) = [d, \iota(\xi)] = d\iota(\xi) + \iota(\xi)d.$$

Another important construct is the following. If $W = W^0 \oplus W^1$ is a graded vector space, then the algebra of endomorphisms $\mathrm{End}(W)$ is also graded. An endomorphism is even if it preserves W^0 and W^1; it is odd if it exchanges them. If we represent endomorphisms as 2×2 matrices, then the even endomorphisms are diagonal matrices and the odd endomorphisms are off-diagonal matrices.

Now there is a device we can use to simplify computations in SuperAlgebra. Namely, we introduce auxiliary odd parameters $\eta_1, \dots, \eta_N$ which anticommute (1.7) and treat them as scalars. In other words, instead of working over $\mathbb{R}$, say, we work over the ring

$$R = \mathbb{R}[\eta_1, \dots, \eta_N].$$

In this way we can work with even elements only and use the sign rule only when commuting these auxiliary parameters η_i past other elements.

We illustrate this general description with the skew-symmetry law (1.8) in a graded Lie algebra $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$. Thus suppose $a, b \in \mathfrak{g}^1$ are odd elements. Then $\eta_1 a$ and $\eta_2 b$ are even elements. The skew-symmetry of the bracket (for even elements) says

$$[\eta_1 a, \eta_2 b] = -[\eta_2 b, \eta_1 a].$$

Now we simplify both sides remembering that we are to treat the η_i as scalars. Thus on the left hand side we bring the η_1 to the left of the bracket and then commute η_2 past a and pick up a minus sign:

$$\mathrm{LHS} = -\eta_1\eta_2[a, b].$$

On the right hand side we first commute η_1 past the even element $\eta_2 b$ (no sign!) and then bring the η_2 in front of the bracket:

$$\mathrm{RHS} = -\eta_1\eta_2[b, a].$$

Combining the last three equations we learn that the bracket is symmetric on odd elements:

$$[a, b] = [b, a], \qquad a, b \text{ odd}.$$

We suggest that the reader try various cases of the Jacobi identity using the auxiliary odd parameters.

A few remarks:

- The odd parameters are introduced in this algebraic context purely as an aid in computation. They allow us to consider only even elements and so use the usual rules of whatever algebraic structure we have at hand. Experience shows that fewer sign mistakes are made this way.
- At a more formal level we can say that a graded Lie algebra $\mathfrak{g}$ is equivalent to a functor

  $$R \longrightarrow (R \otimes \mathfrak{g})^0$$

  from the category of commutative superalgebras to the category of ordinary Lie algebras. The tensor product here is a graded tensor product. Even elements of $R \otimes \mathfrak{g}$ are sums of products (even element of R)$\otimes$(even element of $\mathfrak{g}$) and (odd element of R)$\otimes$(odd element of $\mathfrak{g}$). We have a similar statement for any algebraic structure, not just Lie algebras.
- A variable, but finite, number of these "anticommuting constants" is always available. This matches physicists' use of such constants.
- In this algebraic context these auxiliary parameters are mainly a computational aid, but we will find in a geometric context that they are crucial.

While on the subject of SuperAlgebra it is worth making a few remarks which we will need in later lectures. Our first two remarks are consequences of the sign rule and dictate conventions which are not the usual conventions in the physics literature. This is unfortunate, but since we have a consistent rule of signs in SuperAlgebra it seems dangerous to violate it in special cases. The last remark is a choice of sign convention (among two valid possibilities) which we want to make explicit.

First: Suppose A is an ordinary algebra over $\mathbb{C}$. Then a *conjugation*

$$*\colon A \longrightarrow A$$

is a map which satisfies

$$\begin{aligned} (\lambda a)^* &= \bar{\lambda} a^*, \qquad && \lambda \in \mathbb{C}, \quad a \in A; \\ (a^*)^* &= a, && a \in A; \\ (ab)^* &= b^* a^*, && a, b \in A. \end{aligned} \tag{1.9}$$

Common examples are the complex numbers, the quaternions, and the algebra of complex square matrices (with conjugation being the conjugate transpose). Now if A is a superalgebra over $\mathbb{C}$ then the sign rule demands that (1.9) be replaced by

$$(ab)^* = (-1)^{|a|\,|b|}\, b^* a^*, \qquad a, b \text{ homogeneous.}$$

In particular, if A is commutative, then for all elements $a, b \in A$ we have

$$(ab)^* = a^* b^*, \qquad \text{if } A \text{ commutative.}$$

Note that the product of real elements is real. (An element a is *real* if $a^* = a$.) This comes up in Lecture 5 when we consider the four dimensional superspace $M^{4|4}$.

Our second remark concerns the definition of a super Hilbert space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$. The remark is algebraic, so we can restrict to finite dimensional Hilbert spaces. One of the axioms for the inner product in an ordinary Hilbert space is that it be sesquilinear, and in the super situation this axiom picks up a sign:

$$\overline{\langle v, v' \rangle} = (-1)^{|v|\,|v'|} \langle v', v \rangle. \tag{1.10}$$

Here v, v' are homogeneous elements. Note that the evenness of the inner product implies that $\mathcal{H}^0$ and $\mathcal{H}^1$ are perpendicular. Now (1.10) implies that the norm square of an even element is real and the norm square of an odd element is purely imaginary. We take as positivity requirement the conditions

$$\begin{aligned} \langle v^0, v^0 \rangle &> 0, \\ -i\langle v^1, v^1 \rangle &> 0, \end{aligned}$$

where v^0 is even and v^1 is odd. We leave it to the reader to see from the sign rule that the eigenvalues of a self-adjoint odd operator T lie on the line $i^{1/2}\mathbb{R} \subset \mathbb{C}$. (The eigenvalues of a skew-adjoint operator lie on the line $i^{3/2}\mathbb{R} \subset \mathbb{C}$.)

Finally, we discuss a sign convention. Suppose $W = W^0 \oplus W^1$ is a super vector space with basis $\{e_i, f_\alpha\}$ and dual basis $\{e^i, f^\alpha\}$. Consider the exterior algebra $\bigwedge W^*$. For the even part there is no question that

$$e^i \wedge e^j = -e^j \wedge e^i$$

as usual. The sign comes since the e^i have *cohomological* degree 1. The ambiguity comes when we consider $e^i \wedge f^\alpha$. Since f^α is odd it has odd parity and it is also of odd cohomological degree. So should we consider the total degree to be even? More formally, each element of $\bigwedge W^*$ has both a *parity* p and a *cohomological degree* c. Our choice is the sign rule

$$\omega \wedge \omega' = (-1)^{pp' + cc'} \omega' \wedge \omega, \tag{1.11}$$

where ω has parity p and cohomological degree c and similarly for ω'. (The other valid choice is to use the total degree $p+c$ and so replace the sign in (1.11) with $(-1)^{(p+c)(p'+c')}$. It leads to consistent computations with different signs in the formulas.) Our choice has the consequence

$$\begin{aligned} e^i \wedge f^\alpha &= -f^\alpha \wedge e^i, \\ f^\alpha \wedge f^\beta &= \quad f^\beta \wedge f^\alpha. \end{aligned}$$

(The other sign convention gives a plus sign in the first equation.)

A brief return to quantum particles

Recall that the one particle Hilbert space of a particle can be considered bosonic or fermionic according to how it behaves when we have identical particles. Thus it is natural to collect together all of the particles in a theory and form the *one particle Hilbert space* $\mathcal{H}$ of the theory as the direct sum of the Hilbert spaces of the individual particles in the theory. We write it as a graded space

$$\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$$

where the even part consists of bosons and the odd part consists of fermions. This has the nice consequence that the two particle states are elements of the symmetric square

$$\mathrm{Sym}^2 \mathcal{H} \;\cong\; \mathrm{Sym}^2 \mathcal{H}^0 \quad\oplus\quad \mathcal{H}^0 \otimes \mathcal{H}^1 \quad\oplus\quad \mathrm{Sym}^2 \mathcal{H}^1.$$

The first summand consists of states with two bosons, the second states with one boson and one fermion, and the third states with two fermions. But remember the sign rule! In the symmetric square of the one particle fermion space $\mathcal{H}^1$ we pick up a sign

$$v \otimes v' = -v' \otimes v, \qquad v, v' \in \mathcal{H}^1$$

from commuting the odd elements v, v'. This is precisely the sign λ in (1.2). Thus the sign rule of SuperAlgebra encodes the statistics of a particle.

Now it is clear that the k particle states are elements of $\mathrm{Sym}^k \mathcal{H}$. The *Fock space* of the theory is the sum of all the symmetric powers. Note that the *vacuum* lies in the 0^{th} symmetric power, which is simply the trivial space $\mathrm{Sym}^0 \mathcal{H} \cong \mathbb{C}$. In a *free* quantum field theory this is the entire structure. The Poincaré group acts in the natural way on the Fock space $\mathrm{Sym}^\bullet \mathcal{H}$ induced from the representation on the one particle Hilbert space $\mathcal{H}$. An *interacting* theory has more—an S-matrix which encodes the interactions.

In a relativistic theory the Poincaré symmetries act as even transformations on $\mathcal{H}$. These symmetries map bosons to bosons and fermions to fermions. The same is true for the infinitesimal Poincaré symmetries. *Supersymmetries* at the quantum level are simply odd endomorphisms of $\mathcal{H}$. We will see a simple example at the classical level in Lecture 2. In Lecture 3 we give a general description of the quantum supersymmetry algebras.

Classical particles and their quantization

There is a fairly clean mathematical theory which passes from a *free* classical particle or field to a corresponding free quantum system. This involves quantizing symplectic vector spaces. In this first lecture we only consider classical particles, not classical fields. Our goal here is twofold: First, we want to briefly remind the reader of the crudest ideas of this free quantization. Second, we want to understand how to write a classical model whose quantization leads to fermions. Thus we begin with ordinary (bosonic) classical models.

There are three main ingredients in a classical theory:

- A function space $\mathcal{F}$ of *fields*
- A set of *classical equations of motion* whose solution space is a subset $\mathcal{M} \subset \mathcal{F}$
- A symplectic structure on $\mathcal{M}$.

In Lecture 2 we discuss more systematically the framework of *Lagrangian field theory*. We will see how the classical equations and the symplectic structure are derived from a lagrangian, and how the lagrangian also encodes crucial information about the symmetries in the theory. For today we simply write down these things without deriving them from some more fundamental objects. We should be clear that the symplectic structure is not expected for a field theory on some general curved spacetime, but only when spacetime is written as a product of space and time.

For n dimensional spacetimes the fields are functions on an n dimensional manifold. Today we consider classical nonrelativistic particles which are simply functions of time. So our "spacetime" is just time ($\mathbb{R}$) and $n = 1$. The "space" is the target of the map.

Let $W^0 = \mathbb{R}^m$ denote space. (This 'W' has nothing to do with the 'W's which appeared previously.) We take it to have a standard inner product, which we write as g_{ij} with respect to some fixed basis. The "fields" in this theory are maps from time to space:

$$\mathcal{F}^0 = \{x\colon \mathbb{R} \to W^0\}.$$

We consider a free nonrelativistic particle of mass m. It experiences no force and so Newton's law states that the classical equation of motion is that its acceleration vanish:

$$\ddot{x}(t) = 0.$$

We use the dot notation for time derivatives. The space $\mathcal{M}^0$ of solutions is the set of maps which have constant velocity. This is a vector subspace of $\mathcal{F}^0$. If we fix a time t_0 then we can identify such a map by the position and velocity at that time:

$$\begin{aligned} \mathcal{M}^0 &\longrightarrow W^0 \oplus W^0 \\ x &\longmapsto x(t_0) \oplus \dot{x}(t_0) \end{aligned} \tag{1.12}$$

Here we should view $W^0 \oplus W^0$ as the tangent bundle to W^0. The symplectic structure on this space can be written as

$$\omega = m\langle \delta\dot{x}(t_0) \wedge \delta x(t_0)\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the inner product on W^0 and δ is the differential on $\mathcal{M}$. The symplectic structure is independent of t_0. The notation will become clearer in the next lecture. Using the isomorphism (1.12) we can write it as

$$\omega(v \oplus w, v' \oplus w') = m\{\langle w, v'\rangle - \langle w', v\rangle\}.$$

Note that the two summands W^0 in (1.12) are lagrangian.

We will see in the next lecture that the classical equations and the symplectic form can be derived from the *lagrangian density*

$$L = \frac{m}{2}|\dot{x}(t)|^2\, dt.$$

Quantization assigns to the symplectic vector space $\mathcal{M}^0$ a complex Hilbert space $\mathcal{H}^0$. We would like this assignment to be as functorial as possible. Because quantum states have a phase ambiguity, we might only expect to produce a projective Hilbert space.

Here are two possible interpretations of the functoriality wish:

- We might demand that the symmetries of the symplectic vector space be represented as unitary operators on the Hilbert space. More precisely, we expect a projective representation of the symplectic group. This is usually constructed by taking a splitting

  $$\mathcal{M}^0 \cong L \oplus L'$$

  into the sum of two lagrangian subspaces. With respect to this splitting (called a *polarization*) we define the Hilbert space to be a certain space of L^2 functions on L. With an eye towards the future we single out the dense subspace of polynomial functions, which is the symmetric algebra on the dual space:

  $$\mathcal{H}^0 \supset \mathrm{Sym}^\bullet(L^*) \otimes \mathbb{C}. \tag{1.13}$$

  Of course, this description depends on a choice of lagrangian splitting, and it turns out that the underlying projective space is independent of the splitting. An equivalent statement is that there is a projective unitary representation of the symplectic group, called the *metaplectic representation*, and the quantum Hilbert space is the underlying complex Hilbert of that representation. This is a very satisfactory fulfillment of the functoriality wish. We usually take the lagrangian splitting to be the one in (1.12) and so describe the Hilbert space as the space of functions on W^0. This is the spin 0 nonrelativistic representation we described at the beginning of the lecture.

- Instead of focusing on the points of $\mathcal{M}^0$ and the points of $\mathcal{H}^0$, we might consider the functions on $\mathcal{M}^0$ and the operators on $\mathcal{H}^0$. In the usual way the functions on a symplectic vector space comprise a Poisson algebra. In this case the Poisson bracket on linear functions is
$$\{\dot{x}^i(t), x^j(t)\} = \frac{g^{ij}}{m},$$
where g^{ij} is the inverse metric. (We review Poisson brackets in Lecture 2.) From this perspective quantization should be a map which assigns to each real-valued function on $\mathcal{M}^0$ a self-adjoint operator $\hat{f}$ on $\mathcal{H}^0$. The functoriality wish in this case is that Poisson brackets are mapped into operator brackets:
$$[\hat{f}, \hat{g}] = -i\hbar\{f, g\}\hat{}. \tag{1.14}$$
(Here $\hbar$ is the physical constant which enters into quantum theories.) The the story is less rosy than before. We can indeed represent inhomogeneous quadratic functions according to this wish—that's just the infinitesimal version of the metaplectic representation. But cubic functions and higher cannot be so represented and compromises must be made.

Question: Is there something classical we can quantize to obtain a super Hilbert space? In other words, is there something classical we can quantize to see fermions at the quantum level?

We work backwards. We don't immediately specify the fields and equations, but simply postulate a space of solutions which looks promising. Our first guess is to replace the symplectic vector space $\mathcal{M}^0$ by a graded symplectic vector space. For simplicity we consider an odd symplectic vector space $\mathcal{M}^1$. Recall the sign rule! The skew symmetry of the symplectic form on $\mathcal{M}^1$ means that the form is symmetric on the underlying even vector space. It turns out that we also need to impose a positivity condition, and we record the sign here:
$$\omega(\psi, \psi) < 0, \qquad \psi \in \mathcal{M}^1.$$
As on even symplectic vector spaces we have Poisson brackets, only on two odd elements the Poisson bracket is symmetric. The positivity condition in the language of Poisson brackets is
$$\{\psi, \psi\} < 0, \qquad \psi \in \mathcal{M}^1.$$
Now we imitate our discussion of quantization. In the first approach to quantization we must start by polarizing $\mathcal{M}^1$, i.e., we want to write it as the sum of two maximal isotropic subspaces. But

over the reals there are no proper isotropic subspaces. Thus we complexify and consider complex polarizations. Then we can write

$$\mathcal{M}^1 \otimes \mathbb{C} \cong L_{\mathbb{C}} \oplus L'_{\mathbb{C}}$$

as the sum of complex lagrangians provided that the dimension of $\mathcal{M}^1$ is even. Assuming that, and following the previous (algebraic) prescription (1.13) we form the Hilbert space

$$\mathcal{H} = \mathrm{Sym}^{\bullet}(L^*_{\mathbb{C}}).$$

Since $L_{\mathbb{C}}$ is odd this is a finite dimensional vector space (between friends: an exterior algebra). So in this case the entire Hilbert space has an algebraic description. The reader may recognize this as a standard construction of the spin representation of an orthogonal group in even dimensions. Exactly parallel to the previous story we find that the underlying projective space is canonically independent of the polarization, i.e., the group of symplectic symmetries of the odd vector space $\mathcal{M}^1$ (between friends: an orthogonal group) acts projectively on $\mathcal{H}$. The $\mathbb{Z}/2$ grading is the usual splitting of spinors in even dimensions.

In the second approach we attempt to represent Poisson brackets by operators. If we choose a basis $\{\psi^i\}$ of linear functions on $\mathcal{M}^1$, then the Poisson brackets are

$$\{\psi^i, \psi^j\} = -\frac{g^{ij}}{m} \tag{1.15}$$

for some positive definite bilinear form g. (The positive constant m is put in analogously to the mass in the previous case.) We recognize this as the defining relation of a Clifford algebra, and the quantization problem is to find a $\mathbb{Z}/2$-graded complex Hilbert space on which this Clifford algebra acts by self-adjoint operators with brackets satisfying (1.14). The solution to this problem is again the spin representation (Clifford module). From this point of view we can even make sense of the quantization without assuming that $\mathcal{M}^1$ is even dimensional. Recall, though, that the sign rule in SuperAlgebra forced us into some unconventional conventions about graded Hilbert spaces and self-adjoint operators. These show up here.

Emboldened by our success we now try to construct the other elements of a classical theory—a function space $\mathcal{F}^1$ and an equation of motion which cuts $\mathcal{M}^1$ out of this function space. Here we make a very simple ansatz: Let $W^1 = \mathcal{M}^1$ be the odd vector space we have been considering. Then define $\mathcal{F}^1$ to be the function space

$$\mathcal{F}^1 = \{\psi \colon \mathbb{R} \to W^1\} \tag{1.16}$$

and take the classical equation of motion to be simply

$$\dot{\psi}(t) = 0.$$

In the next lecture we will see that this equation and the symplectic form can be derived from the lagrangian

$$L = \frac{m}{2}\langle \psi(t), \dot{\psi}(t)\rangle \, dt. \tag{1.17}$$

Here $\langle \cdot , \cdot \rangle$ is the symmetric bilinear form whose inverse appears in (1.15).

Be careful! Until the last paragraph we always considered odd vector spaces in a purely algebraic context, and there we just manipulate odd vectors the same way we manipulate even vectors. But in (1.16) we are considering paths in an odd vector space, and so for the first time we are attempting to do geometry on such a space. What does that mean? Certainly we have given no indication so far of what that means, much less ascribed a meaning to (1.17). We resolve these issues now.

Spaces with anticommuting functions

There are two basic approaches to the elementary theory of manifolds: We can focus on the points or dually we can focus on the functions. The latter point of view is more adapted to algebraic geometry, and it is the approach we adopt here. So to describe how to think about the odd vector space W^1 as a manifold, we need to specify its ring of smooth functions. For an even vector space W^0 the ring of smooth functions contains the dense subset of polynomial functions:

$$C^\infty(W^0) \supset \mathrm{Sym}^\bullet\big((W^0)^*\big).$$

So it is natural to *define* the ring of functions on an odd vector space W^1 to be

$$C^\infty(W^1) := \mathrm{Sym}^\bullet\big((W^1)^*\big).$$

Since W^1 is odd this is a finite dimensional algebra, as we have seen many times by now. The use of 'C^∞' in this context is a bit funny, perhaps, but as a differential geometer I like to think by analogy with the situation for smooth manifolds.

Here is the crucial point: The ring of functions $C^\infty(W^1)$ contains a large subring of *nilpotent* elements. In fact,

$$C^\infty(W^1)/\text{nilpotents} \cong \mathbb{R}.$$

Notice that this quotient is simply the ring of functions on a point. This observation is crucial: *The fact that $C^\infty(W^1)$ is not a commutative algebra is a bit of a red herring; more important is the fact that it contains nilpotent elements.* Of course, the ring of functions on a smooth manifold contains no nonzero nilpotents. On the other hand, nilpotent functions are not uncommon in algebraic geometry and we can gain intuition by considering a simple example of a *commutative* ring with nilpotents.

We do not mean to say that the signs in the superalgebra $C^\infty(W^1)$ are not important. They are. But for the basic intuition of how to picture the space W^1 they are less essential than the nilpotency.

Now to the example. It is simply the case where $\dim W^1 = 1$, but we forget about the $\mathbb{Z}/2$-grading. Thus consider a space P whose ring of functions is

$$C^\infty(P) = \mathbb{R}[\epsilon]/(\epsilon^2 = 0).$$

(An algebraic geometer would write $P = \operatorname{Spec}\mathbb{R}[\epsilon]/(\epsilon^2)$.) Now we ask what is a map $P \to M$ for M a smooth manifold. This is opposite to our problem, where we map a smooth manifold ($\mathbb{R}$) into a space with nilpotents (W^1), and we will come to that shortly. In later lectures we will also meet this case of maps from spaces with nilpotents into ordinary manifolds. Now in general a smooth map between manifolds is equivalent to an algebra homomorphism on functions in the opposite direction. So a map $P \to M$ is given by a homomorphism

$$\begin{aligned} C^\infty(M) &\longrightarrow \mathbb{R}[\epsilon]/(\epsilon^2) \\ f &\longmapsto A(f) + B(f)\epsilon. \end{aligned}$$

The condition that this be an algebra homomorphism leads to two equations. The first states that $A\colon C^\infty(M) \to \mathbb{R}$ is an algebra homomorphism. The only such homomorphisms are evaluation at some point $m \in M$:

$$A(f) = f(m).$$

The second equation states that B is a derivation over functions, so is given as directional derivative in the direction of some tangent vector $\xi_m \in T_mM$:

$$B(f) = \xi_m f.$$

So how might we picture P? It is in some sense an abstract tangent vector. We could represent it as an arrow (Figure 4). Or we might think of it as some sort of thickened point, a point together with a "nilpotent cloud" or "nilpotent fuzz" surrounding it. We should think that the nilpotent cloud has a definite shape, but in more complicated examples that would be hard to represent in detail pictorially. The reader may wish to consider the space whose ring of functions is $\mathbb{R}[\epsilon]/(\epsilon^n)$

Figure 4: Two renditions of the space P

for larger n, or the space whose ring of functions is $\mathbb{R}[\epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_2^2)$ with everything commutative. Geometrically each of these examples is a point with some nilpotent cloud around it.

Now what about a map in the other direction, a map $M \to P$? Such a map is given by a homomorphism $\mathbb{R}[\epsilon]/(\epsilon^2) \to C^\infty(M)$. Now since the ring of functions on a smooth manifolds has no nilpotents, the image of ϵ is necessarily zero. In other words, any map $M \to P$ simply maps the whole manifold M to a point, the geometric point in P.

We apply this discussion to our situation:

- As a manifold we picture the odd vector space W^1 as a point with a nilpotent cloud surrounding it. By contrast with the motivating example P, the cloud surrounding W^1 has anticommuting functions on it, but the geometric picture is essentially the same. Certainly the drawing is the same.
- Any map $\psi\colon \mathbb{R} \to W^1$ is identically zero!

This last conclusion is not what we want, since we'd like to give meaning to odd functions and use them in classical lagrangians. In other words, we'd like to be able to probe the nilpotent cloud in W^1.

Here we use another idea from algebraic geometry, Grothendieck's "functor of points". The idea is that an ordinary manifold M cannot probe the fuzz in a target, but we need to use manifolds with fuzz to probe the fuzz. In our situation we are studying maps $\mathbb{R} \to W^1$, and so we introduce nilpotents in the domain by studying families of maps parametrized by a space with nilpotents. Thus let B the the space whose ring of functions is

$$C^\infty(B) = \mathbb{R}[\eta_1, \dots, \eta_N], \qquad \eta_i\eta_j = -\eta_j\eta_i.$$

This is the commutative superalgebra we used before as a computational aid in SuperAlgebra. Here we again introduce the ηs as auxiliary parameters, but now in the guise of a space B. Geometrically B is a point with a cloud of "superfuzz". We consider the product space $B \times \mathbb{R}$, which has a ring of functions

$$C^\infty(B \times \mathbb{R}) = \mathbb{R}[\eta_1, \dots, \eta_N] \otimes C^\infty(\mathbb{R}).$$

Again we treat the ηs as constants; geometrically we are working with a family of spaces parametrized by the base B. (See Figure 5.)

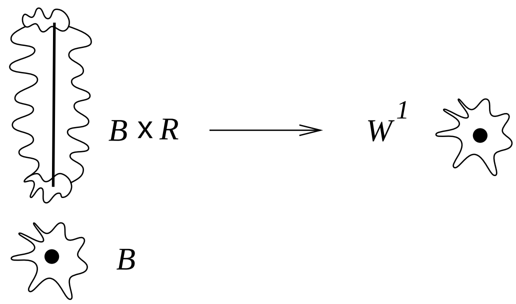

Figure 5: The family $B \times \mathbb{R}$ and a map to W^1

Now we can have nontrivial maps $B \times \mathbb{R} \to W^1$. To illustrate let's start with a single η so that $C^\infty(B) = \mathbb{R}[\eta]$. Choose a basis $\{\psi^i\}$ of $(W^1)^*$; then

$$C^\infty(W^1) = \mathbb{R}[\psi^1, \dots, \psi^m], \qquad \psi^i\psi^j = -\psi^j\psi^i.$$

We suppose that the inverse metric in this basis is $g^{ij} = \langle \psi^i, \psi^j \rangle$. Then a map $\psi\colon B \times \mathbb{R} \to W^1$ is given as

$$\begin{aligned} C^\infty(W^1) &\longrightarrow \mathbb{R}[\eta] \otimes C^\infty(\mathbb{R}) \\ \psi^i &\longmapsto a^i(t)\eta \end{aligned} \tag{1.18}$$

for some functions $a^i \in C^\infty(\mathbb{R})$. Remember our postulate that when we introduce auxiliary parameters all of our structures are even. Here this means that the algebra homomorphism (1.18) is even, and so it maps the odd element ψ^i to an odd element of the codomain. With the odd parameter we have nonzero odd elements of the codomain, and so nonzero maps. We write

$$\psi^i(t) = a^i(t)\eta. \tag{1.19}$$

As advertised, ψ is a family of maps parametrized by B, and the formula makes clear that at the geometric point of B ($\eta = 0$) the map vanishes. This is what we found before we introduced the odd parameters.

Finally, we can answer the question posed at the beginning of the lecture and ascribe meaning to (1.17). If we simply plug (1.19) into (1.17) we find that L still vanishes. To get a nonvanishing answer we must go to a base space with two ηs. In that case we have

$$\psi^i(t) = f^i(t)\eta_1 + g^i(t)\eta_2,$$

and we compute

$$L = \Big[\frac{m}{2} g_{ij} \{ a^i(t) \dot b^j(t) - b^i(t) \dot a^j(t) \}\; dt \Big]\; \eta_1 \eta_2. \tag{1.20}$$

This is nonzero in the family, but as expected vanishes at the geometric point $\eta_1 = \eta_2 = 0$. The *action* S is the integral of the lagrangian, which here means the integral over the fibers of the projection $B \times \mathbb{R} \to B$:

$$S = \Big[\int_{\mathbb{R}} \frac{m}{2} g_{ij} \{ a^i(t) \dot b^j(t) - b^i(t) \dot a^j(t) \}\; dt \Big]\; \eta_1 \eta_2.$$

(We assume our functions to have compact support.) Thus the action is a function on B, an element of $C^\infty(B)$. It is even, as we expect. (The lagrangian density (1.20) is even as well.) We might have made the same construction in the case of a classical bosonic particle $x\colon \mathbb{R} \to W^0$. We could have looked at a family of paths parametrized by an ordinary smooth manifold B, and we would have found the action to be a function on B. In that situation such families carry no more information than a single path. But when we have odd variables the families are essential to have nontrivial maps.

For fun the reader should write the analog of (1.20) with three ηs. Now there is an $\eta_1\eta_2\eta_3$ term in $\psi^i(t)$ and the formulas are more complicated. The functoriality of these computations is clear: Set $\eta_3 = 0$ to recover (1.20).

A few concluding remarks:

- More formally, we work functorially over bases B whose ring of functions is a commutative superalgebra. In a given problem we fix a base B, then do all of the computations. Here that means define the space of fields, compute the lagrangian, derive the equations of motion, classical space of solutions, symplectic structure, etc. Then we quantize to obtain a Hilbert space with coefficient ring $C^\infty(B)$. So at the end of the day we produce a Hilbert space functorially from a commutative superalgebra. As we discussed earlier, this is equivalent to a graded Hilbert space.
- The odd parameters were simply a computational device in SuperAlgebra. In SuperGeometry they are essential for the geometric interpretation.
- Manifolds with superfuzz are more commonly known as *supermanifolds*, and we will of course refer to them as such.
- We will not make explicit the auxiliary odd parameters and the base spaces again in these lectures. The reader is well-advised to put them into some of the computations to better see the geometric pictures.
- We will not give a formal development of SuperGeometry and supermanifolds in these lectures. We will simply compute with them as we do in ordinary geometry. Thus we have

tangent vectors, differential forms, bundles and connections, etc. We treat the nilpotent cloud as if it were a geometric direction. Hopefully the discussion here provides the necessary intuition and some idea as to how a more rigorous development would go. One imitates a development of ordinary differential geometry which emphasizes the functions rather than the points.

- The picture we have of "classical fermions" is in accord with the physical intuition. In some sense there are no classical fermions, which is the statement that the geometric points do not see the anitcommuting functions. On the other hand we can picture them as living in a nilpotent cloud and so use our usual geometric ideas. When we pass to the algebraic situation of the quantum theory they are on an equal footing with the bosons.

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