ESSENTIAL KILLING FIELDS OF PARABOLIC GEOMETRIES: PROJECTIVE AND CONFORMAL STRUCTURES

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Abstract. We use the general theory developed in our article [1] in the setting of parabolic geometries to reprove known results on special infinitesimal automorphisms of projective and conformal geometries.

1. Introduction

This text is a complement to our article [1] which studies special infinitesimal automorphisms in the general setting of parabolic geometries. We illustrate the general theory developed there by reproving the known results on such automorphisms in the cases of projective structures (see [5]) and of conformal structures (see [4] and [3]). The main moral of [1] is that special infinitesimal automorphisms can be understood to a large extent by doing purely algebraic computations on the level of the Lie algebra which governs the geometry in question. The output of these computations can then be nicely interpreted geometrically, providing examples of the powerful interplay between algebra and geometry for the class of parabolic geometries. Even for well known geometries as the two examples discussed here, this leads to precise new descriptions of the behavior of these special flows.

The two examples of structures discussed in this article belong to the subclass of parabolic geometries related to so–called $|1|$–gradings. These geometries have been studied under the names AHS–structures, irreducible parabolic geometries, and abelian parabolic geometries, in

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the literature. We will restrict all discussions in this article to geometries in this subclass, referring to [1] for more general concepts.

The notion of higher order fixed point, which is the main concept studied in [1], becomes very simple in the AHS–case: A point x_0 is a *higher order fixed point* of an infinitesimal automorphism η of an AHS–structure if and only if the local flow φ^t of η fixes x_0 to first order, meaning $\varphi^t(x_0) = x_0$ and $D\varphi^t(x_0) = \mathrm{Id}$ for all t (see [1, Def. 1.5] for the general definition). Infinitesimal automorphisms with this property exist on the homogeneous model of each AHS–structure (see 2.2 below) and, as observed in [1, Rmk. 1.6], they are always *essential*. For the structures treated in this article, the latter condition simply means not preserving any affine connection in the projective class, respectively, any metric in the conformal class.

The basic question is to what extent infinitesimal automorphisms admitting a higher order fixed point can exist on non–flat geometries. The simplest possible answer to this question would be that existence of a higher order fixed point implies local flatness on an open neighborhood of this fixed point, which happens for projective structures. In this case, the infinitesimal automorphism in question is conjugate via a local isomorphism to an infinitesimal automorphism on the homogeneous model of the geometry which has a higher order fixed point, and the latter can be explicitly described.

In general, and already in the case of conformal pseudo-Riemannian structures, the situation is less simple. The next best case is that existence of a higher order fixed point x_0 implies local flatness on an open subset U such that $x_0 \in \overline{U}$. For example, for higher order fixed points of conformal flows of timelike or spacelike type (see 3.2 below), the set U is the interior of the light cone, or the interior of its complement, respectively, intersected with a neighborhood of x_0. It seems difficult to precisely describe the flow outside of U, but one can obtain detailed information about the flow on $\overline{U}$ from the results of [1] (or from the results of [5] and [4] in the projective and conformal cases, respectively).

2. Background

In this section, we very briefly review some background on AHS–structures referring to sections 3.1, 3.2, and 4.1 of [2], and we collect the results from [1] we will need in the sequel.

2.1. **AHS–structures.** The basic data needed to specify an AHS–structure is a semisimple Lie algebra $\mathfrak{g}$ endowed with a $|1|$–grading, a decomposition $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ making $\mathfrak{g}$ into a graded Lie algebra. This means that $\mathfrak{g}_{\pm 1}$ are abelian Lie subalgebras, while $\mathfrak{g}_0$ is a Lie subalgebra which acts on $\mathfrak{g}_{\pm 1}$ via the restriction of the adjoint action. The only additional information encoded in $\mathfrak{g}$ is the restriction of the Lie bracket of $\mathfrak{g}$ to a map $[\ ,\ ] : \mathfrak{g}_1 \otimes \mathfrak{g}_{-1} \to \mathfrak{g}_0$. It turns out that $\mathfrak{g}_0$ is always reductive, and its action on $\mathfrak{g}_{-1}$ defines a faithful representation. Finally $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a maximal parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{g}_1$ is an ideal in $\mathfrak{p}$. The Killing form of $\mathfrak{g}$ induces an isomorphism $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{g}_1$ of $\mathfrak{p}$–modules, which can also be interpreted as an isomorphism $(\mathfrak{g}_{-1})^* \cong \mathfrak{g}_1$ of $\mathfrak{g}_0$–modules.

Choosing a Lie group G with Lie algebra $\mathfrak{g}$, it turns out that the normalizer $N_G(\mathfrak{p})$ is a closed subgroup of G with Lie algebra $\mathfrak{p}$. One next chooses a *parabolic subgroup* $P \subset G$ corresponding to $\mathfrak{p}$, a subgroup lying between $N_G(\mathfrak{p})$ and its connected component of the identity. Then one defines $G_0 \subset P$ to be the closed subgroup consisting of all elements whose adjoint action preserves the grading of $\mathfrak{g}$. This subgroup has Lie algebra $\mathfrak{g}_0$ and it naturally acts on $\mathfrak{g}_{-1}$ via the adjoint action. Finally, the exponential mapping restricts to a diffeomorphism from $\mathfrak{g}_1$ onto a closed normal subgroup $P_+ \subset P$ and P is the semidirect product $G_0 \ltimes P_+$.

An AHS–structure of type $(\mathfrak{g}, P)$ on a smooth manifold M which has the same dimension as G/P is then defined as a *Cartan geometry* of that type, which consists of a principal P–bundle $p : B \to M$ and a *Cartan connection* $\omega \in \Omega^1(B, \mathfrak{g})$. The Cartan connection trivializes the tangent bundle TB, is equivariant with respect to the principal right action of P, and reproduces the generators of fundamental vector fields (see [2, Sec. 1.5] for the definition). The *homogeneous model* of the geometry is the bundle $G \to G/P$ with the left-invariant Maurer–Cartan form as the Cartan connection.

Via the Cartan connection, TM can be identified with the associated bundle $B \times_P (\mathfrak{g}/\mathfrak{p})$. More precisely, one can form $B_0 := B/P_+$ which is a principal bundle over M with structure group $P/P_+ \cong G_0$, and ω descends to a soldering form on this bundle. Hence one can interpret $B_0 \to M$ as a first order structure with structure group G_0. In this interpretation, $TM \cong B_0 \times_{G_0} \mathfrak{g}_{-1}$, and consequently $T^*M \cong B_0 \times_{G_0} \mathfrak{g}_1 \cong B \times_P \mathfrak{g}_1$.

In all cases except the one corresponding to projective structures, requiring a normalization condition on the curvature of ω makes the Cartan geometry (B, ω) equivalent, in the categorical sense, to the underlying first order structure B_0. In the projective case, this underlying structure contains no information and the Cartan geometry is equivalent to the choice of a projective class of torsion free linear connections on the tangent bundle TM.

For any Cartan geometry, the curvature of a Cartan connection is a complete obstruction to local isomorphism to the homogeneous model, or *local flatness* (see [2, Prop. 1.5.2]). In the case of AHS–structures, there is a conceptual way to extract parts of this curvature, called *harmonic curvature* components, which still form a complete obstruction to local flatness (see [2, Thm. 3.1.12]). In the examples we are going to discuss, there is only one harmonic curvature component, and this is either a version of Weyl curvature or of a Cotton–York tensor.

2.2. **Higher order fixed points.** An infinitesimal automorphism of a Cartan geometry $(B \to M, \omega)$ of type $(\mathfrak{g}, P)$ is a vector field $\tilde{\eta} \in \mathfrak{X}(B)$ which is P–invariant and satisfies $\mathcal{L}_{\tilde{\eta}}\omega = 0$, where $\mathcal{L}$ denotes the Lie derivative. Any P–invariant vector field $\tilde{\eta}$ on B is projectable to a vector field $\eta \in \mathfrak{X}(M)$. From the equivalence to underlying structures discussed in the previous section, one concludes that, apart from the projective case, $\tilde{\eta}$ is an infinitesimal automorphism of the Cartan geometry if and only if η is an infinitesimal automorphism of the underlying first order structure. There is an analogous correspondence in the projective case.

Via the Cartan connection ω, a P–invariant vector field $\tilde{\eta}$ on B corresponds to a P–equivariant function $f : B \to \mathfrak{g}$. Then $\tilde{\eta}$, or the underlying vector field η on M, has a higher order fixed point at $x_0 \in M$ if and only if for one—or equivalently, any—point $b_0 \in B$ with $p(b_0) = x_0$, we have $f(b_0) \in \mathfrak{g}_1 \subset \mathfrak{g}$. (Observe that x_0 is fixed by the flow, or is a zero of η, if and only if $f(b_0) \in \mathfrak{p} \subset \mathfrak{g}$.)

If $\tilde{\eta}$ has a higher order fixed point at x_0, then for each $b_0 \in B$ with $p(b_0) = x_0$ the value $f(b_0) \in \mathfrak{g}_1$ via b_0 corresponds to an element of $T^*_{x_0}M$ since $T^*M \cong B \times_P \mathfrak{g}_1$. Equivariance of f implies that this element does not depend on b_0, so $\tilde{\eta}$ gives rise to a well defined element $\alpha \in T^*_{x_0}M$, called the *isotropy* of $\tilde{\eta}$ (or of η) at the higher order fixed point x_0 [1, Def. 1.5].

The P–orbit $f(p^{-1}(x_0)) \subset \mathfrak{g}_1$ similarly depends only on α. This P–orbit is called the *geometric type* of the isotropy α [1, Sec. 1.2], and is the most basic invariant of a higher order fixed point. In all examples studied here and in [1], different geometric types are discussed separately.

It is possible to do precise calculations for infinitesimal automorphisms with higher order fixed points on the homogeneous model G/P of the geometry. Here the automorphisms are exactly the left translations by elements of G, so infinitesimal automorphisms are right invariant vector fields on G. By homogeneity, we may assume that our automorphism fixes the point $o = eP \in G/P$. Thus the value of an infinitesimal automorphism at o can be naturally interpreted as the generator $Z \in \mathfrak{g}$ of the right invariant vector field in question. Now of course o is a fixed point if and only if $Z \in \mathfrak{p}$ and a higher order fixed point if and only if $Z \in \mathfrak{g}_1$. Hence the flows of such infinitesimal automorphisms are just the left translations by e^{tZ} for $Z \in \mathfrak{g}_1$.

Given a higher order fixed point x_0 of an infinitesimal automorphism η and a neighborhood U of x_0 in M, we call the *strongly fixed component* of x_0 in U [1, Def. 2.3] the set of all points in U which can be reached from x_0 by a smooth curve lying in U, all of whose points are higher order fixed points of the same geometric type as x_0. The higher order fixed point x_0 is called *smoothly isolated* if its strongly fixed component consists of $\{x_0\}$ only.

2.3. **Results from** [1]**.** The first step to apply the results of [1] is to associate to an element $\alpha \in T^*_{x_0}M$ three geometrically significant subsets of the tangent space $T_{x_0}M$. These sets are defined algebraically, after choosing an element $b_0 \in B$ with $p(b_0) = x_0$. Via b_0, the covector α corresponds to an element $Z \in \mathfrak{g}_1$, and the subsets of $\mathfrak{g}_{-1}$ are defined by

$$C(Z) := \{X : [X, Z] = 0\} \subset \{X : [X, [X, Z]] = 0\} =: F(Z)$$
$$T(Z) := \{X : [[Z, X], X] = -2X, [[Z, X], Z] = 2Z\}$$

In [1] the sets were denoted by $C_{\mathfrak{g}_-}(Z)$, and similarly for F and T. Moreover, the original definition of $F_{\mathfrak{g}_-}(Z)$ used there is different, but equivalence to the one used here is observed in the proof of Proposition 2.16 of [1].

For $X \in T(Z)$, we can consider $A := [Z, X] \in \mathfrak{g}_0$. Given a representation $\mathbb{W}$ of G_0 on which A acts diagonalizably, we define $\mathbb{W}_{ss}(A) \subset \mathbb{W}_{st}(A) \subset \mathbb{W}$ as the sum of all eigenspaces corresponding to negative eigenvalues, respectively, to non–positive eigenvalues of A.

Identifying $\mathfrak{g}_{-1} \cong \mathfrak{g}/\mathfrak{p}$ as vector spaces, one can use the element b_0 to identify $C(Z) \subset F(Z)$ and $T(Z)$ with subsets $C(\alpha) \subset F(\alpha)$ and $T(\alpha)$ of $T_{x_0}M$. Equivariance of ω then easily implies that the latter subsets are independent of the choice of b_0, so they are intrinsically associated to $\alpha \in T^*_{x_0}M$. Observe that $C(\alpha)$ is a linear subspace of $T_{x_0}M$, $F(\alpha)$ is only closed under multiplication by scalars, while $T(\alpha)$ is just a subset.

The most concise way to formulate the results of [1] for our purposes is via normal coordinates centered at a point $x_0 \in M$, see [1, Sec. 1.1.2]. For $X \in \mathfrak{g}_{-1}$ we can take the "constant vector field" $\tilde{X} \in \mathfrak{X}(B)$ characterized by $\omega(\tilde{X}) = X$. Write $\varphi^t_{\tilde{X}}(b_0)$ for the flow line of such a field emanating from $b_0 \in B$ with $p(b_0) = x_0$; then we can use $X \mapsto p(\varphi^1_{\tilde{X}}(b_0))$ to define a diffeomorphism from an open neighborhood of 0 in $\mathfrak{g}_{-1}$ onto an open neighborhood of x_0 in M. Combining the inverse of such a diffeomorphism with the identification of $\mathfrak{g}_{-1}$ with $T_{x_0}M$ provided by b_0, we obtain a *normal coordinate chart* centered at x_0 with values in $T_{x_0}M$. Varying b_0 gives a family of charts parametrized by the fiber $p^{-1}(x_0) \cong P$.

Now the first result we need concerns the form of the flow, and in particular further fixed points of an infinitesimal automorphism admitting one higher order fixed point. It is proved in Propositions 2.5 and 2.17 of [1].

Proposition 2.1. *Let η be an infinitesimal automorphism of an AHS–structure having a higher order fixed point at $x_0 \in M$ with isotropy $\alpha \in T_{x_0}M$.*

(1) If $C(\alpha) = 0$, then the higher order fixed point x_0 is smoothly isolated.

(2) In any normal coordinate chart centered at x_0 with values in a neighborhood $U \subset T_{x_0}M$ of zero we have

- (a) *Any point of $U \cap F(\alpha)$ is a zero of η.*
- (b) *Any point of $U \cap C(\alpha)$ is a higher order fixed point of the same geometric type as x_0.*
- (c) *For any $\xi \in T(\alpha)$ the flow on the intersection $U \cap \mathbf{R} \cdot \xi$ is given by $\varphi^t(s\xi) = \frac{s}{1+st}\xi$ for $ts > 0$.*

The second result concerns information on local flatness coming from elements of $T(\alpha)$. This is proved in Corollary 2.14 and Proposition 2.15 of [1].

Proposition 2.2. *Let η be an infinitesimal automorphism of an AHS–structure having a higher order fixed point at $x_0 \in M$ with isotropy $\alpha \in T_{x_0}M$. Suppose that for some element $Z \in \mathfrak{g}_1$ belonging to the geometric type of α and all $X \in T(Z)$, the element $A = [Z, X]$ acts diagonalizably on $\mathfrak{g}_{-1}$ and on each representation $\mathbb{W}$ of G_0 in which one of the harmonic curvature components of the geometry in question has its values. Suppose further that the following conditions are satisfied:*

(1) *For each X, all eigenvalues of A on $\mathfrak{g}_{-1}$ are non–positive and the 0–eigenspace coincides with $C(Z)$.*
(2) *For each X, $\mathbb{W}_{ss}(A) = 0$.*
(3) $\cap_{X \in T(Z)} \mathbb{W}_{st}(A) = 0$.

Then in each normal coordinate chart centered at x_0 with values in a neighborhood $U \subset T_{x_0}M$ of zero and for each $\xi \in T(\alpha)$, there is an open neighborhood of $(\mathbf{R} \cdot \xi) \cap (U \setminus 0)$ on which the geometry is locally flat. In particular, we get local flatness on an open subset containing x_0 in its closure.

3. Results

3.1. **Projective structures.** A projective structure on a smooth manifold M of dimension $n \geq 2$ is given by an equivalence class of torsion free linear connections on TM which share the same geodesics up to parametrization. An infinitesimal automorphism in this case can be simply defined as a vector field η on M whose flow preserves this class of connections or equivalently the family of geodesic paths. What we will prove here is

Theorem 3.1 (compare [5]). *Let $(M, [\nabla])$ be a smooth manifold of dimension $n \geq 2$ endowed with a projective structure and suppose that $\eta \in \mathfrak{X}(M)$ is an infinitesimal automorphism of the projective structure which has a higher order fixed point at $x_0 \in M$. Then there is an open neighborhood of $x_0 \in M$ on which the projective structure is locally flat.*

To describe projective structures as AHS–structures, take $G = PSL(n+1, \mathbf{R})$ and let $P \subset G$ be the stabilizer of a point for the canonical action of G on $\mathbf{RP}^n$. On the level of Lie algebras, $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbf{R})$, and the $|1|$–grading satisfies $\mathfrak{g}_0 \cong \mathfrak{gl}(n, \mathbf{R})$, $\mathfrak{g}_{-1} \cong \mathbf{R}^n$, and $\mathfrak{g}_1 \cong \mathbf{R}^{n*}$. The grading comes from a block decomposition with blocks of sizes 1 and n of the form

$$\begin{pmatrix} -\mathrm{tr}(A) & Z \\ X & A \end{pmatrix}$$

with $X \in \mathbf{R}^n$, $A \in \mathfrak{gl}(n, \mathbf{R})$, and $Z \in \mathbf{R}^{n*}$.

Note $G_0 \cong GL(n, \mathbf{R})$. It follows that there is just one non–zero P–orbit in $\mathfrak{g}_1$, and hence just one possible geometric type of isotropy. The algebra needed for our purpose is in the following lemma:

Lemma 3.2. *For* $0 \neq Z \in \mathfrak{g}_1$*,*

$$0 = C(Z) \subset F(Z) = \{X \in \mathfrak{g}_{-1} : ZX = 0\}$$
$$T(Z) = \{X \in \mathfrak{g}_{-1} : ZX = 1\}$$

Moreover, Z *satisfies all conditions of Proposition 2.2.*

Proof: For $Z \in \mathfrak{g}_1$ and $X, Y \in \mathfrak{g}_{-1}$, the brackets relevant for our purposes are

$$\begin{aligned} [Z, X] &= -XZ \\ [[Z, X], Y] &= -ZYX - ZXY \\ [[Z, X], Z] &= 2ZXZ. \end{aligned}$$

The descriptions of $C(Z)$, $F(Z)$, and $T(Z)$ follow easily.

For $X \in T(Z)$, setting $A = [Z, X]$ gives $[A, Y] = -Y - ZYX$ for all $Y \in \mathfrak{g}_{-1}$. Hence the eigenspace decomposition of $\mathfrak{g}_{-1}$ is $\mathbf{R} \cdot X \oplus \ker(Z)$ with corresponding eigenvalues -2 and -1. In particular, condition (1) of Proposition 2.2 holds.

Now it is well known that for projective structures, there always is only one harmonic curvature component, see Section 4.1.5 of [2]. If $n > 2$, the harmonic curvature is the so–called *Weyl curvature*, the totally tracefree part of the curvature of any connection in the projective class. The corresponding representation $\mathbb{W}$ of $\mathfrak{g}_0$ is contained in $\Lambda^2\mathfrak{g}_1 \otimes \mathfrak{sl}(\mathfrak{g}_{-1})$. From the calculations for $\mathfrak{g}_-$, one can see that the possible eigenvalues of A on $\Lambda^2\mathfrak{g}_1$ are 2 and 3, while the possible eigenvalues on $\mathfrak{sl}(\mathfrak{g}_{-1})$ are -1, 0, and 1. Hence $\mathbb{W}_{st}(A) = 0$ for $n > 2$. For $n = 2$ the basic invariant is a projective analog of the Cotton–York tensor, and the corresponding representation $\mathbb{W}$ is contained in $\Lambda^2\mathfrak{g}_1 \otimes \mathfrak{g}_1$. Clearly, A has only positive eigenvalues on this representation, so $\mathbb{W}_{st}(A) = 0$ holds for $n \geq 2$. The remaining conditions from Proposition 2.2 follow. $\diamondsuit$

Proof: [of Theorem 3.1] The description of $T(Z)$ in the lemma shows that the set of non–zero scalar multiples of elements of $T(\alpha) \subset T_{x_0}M$ is the complement of the hyperplane $\ker(\alpha)$. In particular, the intersection of this set with the range U of a normal coordinate chart is a dense subset of U. Now by Proposition 2.2, the geometry is locally flat on this dense subset and hence on all of U. $\diamondsuit$

3.2. **Pseudo–Riemannian conformal structures.** Recall that two pseudo–Riemannian metrics g and $\hat{g}$ are conformally equivalent if there is a positive smooth function f such that $\hat{g} = f^2 g$. Evidently, conformally equivalent metrics have the same signature. A conformal equivalence class of metrics on a smooth manifold M is called a *conformal structure* on M. This can be equivalently described as a first order structure corresponding to $CO(p,q) \subset GL(p+q,\mathbf{R})$. There are three orbits of $CO(p,q)$ on $\mathbf{R}^{(p+q)*}$, when $0 < p \leq q$. These orbits give three *geometric types* (see [1, Sec 1.2]): spacelike, null, or timelike, and correspond to the sign of the inner product of the vector with itself. For definite signature, there is just one possible geometric type. As we shall see, flows with isotropy equal a spacelike or timelike element of $\mathbf{R}^{p,q*}$ behave very similarly, so the main distinction is between null isotropy and non–null isotropy. Of course, in definite signature, only non–null isotropy is possible. We are going to prove the following.

Theorem 3.3 (compare [3], [4])**.** *Let $(M,[g])$ be a smooth manifold of dimension ≥ 3 endowed with a conformal structure, and let η be an infinitesimal conformal isometry of M. Then higher order fixed points with non–null isotropy are smoothly isolated, while for null isotropy there is a smooth curve contained in the strongly fixed component. Moreover, for any higher order fixed point x_0 of η, there is an open subset $U \subset M$ with $x_0 \in \overline{U}$ on which M is locally conformally flat.*

This theorem is a consequence of the detailed descriptions of infinitesimal automorphisms with the two kinds of higher order fixed points in Propositions 3.5 and 3.7, respectively.

The description of pseudo–Riemannian conformal structures in dimension $n \geq 3$ as parabolic geometries is well known, see Sections 1.6 and

4.1.2 of [2]. A structure of signature (p,q), is modeled on G/P with $G = PO(p+1,q+1)$ and P the stabilizer of a null line in $\mathbf{R}^{p+1,q+1}$. The homogeneous model is the space of null lines in $\mathbf{R}^{p+1,q+1}$, a quadric in $\mathbf{RP}^{n+1}$. The Lie algebra $\mathfrak{g} = \mathfrak{so}(p+1,q+1)$ can be realized as

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & Z & 0 \\ X & A & -\mathbb{I}Z^t \\ 0 & -X^t\mathbb{I} & -a \end{pmatrix} : \begin{array}{c} a \in \mathbf{R}, \\ X \in \mathbf{R}^n,\ Z \in \mathbf{R}^{n*}, \\ A \in \mathfrak{so}(p,q) \end{array} \right\}$$

Here $\mathbb{I}$ is the diagonal matrix $\mathrm{Id}_p \oplus -\mathrm{Id}_q$. The grading corresponding to P has the form $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with the components represented by X, (A,a) and Z, respectively. The relevant bracket formulae for $Z \in \mathfrak{g}_1$ and $X, Y \in \mathfrak{g}_{-1}$, are:

$$\begin{aligned} [Z,X] &= (-XZ + \mathbb{I}(XZ)^t\mathbb{I}, ZX) \\ [[Z,X],Z] &= 2ZXZ - Z\mathbb{I}Z^tX^t\mathbb{I} = 2ZXZ - \langle Z,Z\rangle X^t\mathbb{I} \\ [[Z,X],Y] &= -XZY + \mathbb{I}Z^tX^t\mathbb{I}Y - ZXY \\ &= -XZY - ZXY + \langle X,Y\rangle \mathbb{I}Z^t. \end{aligned} \tag{1}$$

where $\langle\ ,\ \rangle$ denotes the (standard) inner product of signature (p,q) on $\mathfrak{g}_{\pm1}$ corresponding to the matrix $\mathbb{I}$. This leads to $G_0 \cong CO(p,q)$ and $P \cong CO(p,q) \ltimes \mathbf{R}^{n*}$.

In all dimensions $n \neq 4$, there is just one harmonic curvature component. For $n \geq 5$, this component is the Weyl curvature, the totally tracefree part of the Riemann curvature tensor. It is a section of the bundle associated to the irreducible component of highest weight in $\Lambda^2\mathfrak{g}_1 \otimes \mathfrak{so}(\mathfrak{g}_{-1})$. For $n = 3$, the harmonic curvature is the Cotton–York tensor, which is a section of the bundle associated to the irreducible component of highest weight in $\Lambda^2\mathfrak{g}_1 \otimes \mathfrak{g}_1$. In dimension $n = 4$, the Weyl curvature splits into two components according to the splitting of $\Lambda^2\mathfrak{g}_1$ into a self–dual and an anti–self–dual part (with respect to the Hodge–$*$–operator), and these comprise the harmonic curvature.

3.2.1. *Non–null isotropy.* This lemma collects the needed algebraic results. The *grading element* of a Lie algebra $\mathfrak{g}$ with AHS-structure is A

in the center of $\mathfrak{g}_0$ for which $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are the 1- and (-1)-eigenspaces of $\mathrm{ad}(A)$, respectively.

Lemma 3.4. *For a non–isotropic element $Z \in \mathfrak{g}_1$,*

(1) *The sets associated to Z are*

$$0 = C(Z) \subset F(Z) = \{X \in \mathfrak{g}_{-1} : ZX = \langle X, X\rangle = 0\}$$

$$T(Z) = \left\{ \frac{2}{\langle Z, Z\rangle} \mathbb{I}Z^t \right\}$$

(2) *For the unique element $X \in T(Z)$, the bracket $A = [Z, X] \in \mathfrak{g}_0$ is twice the grading element of $\mathfrak{g}$. In particular, all conditions of Proposition 2.2 are satisfied by Z.*

Proof: Assume $[Z, X] = 0$. Then from the brackets in the displayed equations (1) above, $ZX = 0$ and $[[Z, X], Z] = 2ZXZ - \langle Z, Z\rangle X^t\mathbb{I} = 0$. Since $\langle Z, Z\rangle \neq 0$ by assumption, we conclude $X = 0$, so $C(Z) = 0$. Next, suppose that $[[Z, X], X] = 0$. Then formula (1) gives $2ZXX = \langle X, X\rangle \mathbb{I}Z^t$, which is impossible if X and $\mathbb{I}Z^t$ are linearly dependent, so the description of $F(Z)$ follows.

Next, if $[[Z, X], Z]$ is a multiple of Z, then $X^t\mathbb{I}$ must be a multiple of Z. It is easy to compute this multiple, which yields $T(Z)$, and to verify that $A = [Z, X]$ is twice the grading element. This implies that A always acts diagonalizably on all representations of $\mathfrak{g}_0$, and it acts by multiplication by -2 on $\mathfrak{g}_{-1}$. Since all eigenvalues of the grading element on $\Lambda^2\mathfrak{g}_1 \otimes \mathfrak{g}$ are positive, we conclude that $\mathbb{W}_{st}(A) = 0$ for any representation $\mathbb{W}$ corresponding to a harmonic curvature component. $\diamondsuit$

From the general theory of parabolic geometries it follows that, given any normal coordinate chart centered at x_0, straight lines through zero in $T_{x_0}M$ correspond to distinguished curves of the geometry emanating from x_0 (see [2, Sec 5.3]). In conformal geometry, these curves are conformal circles in non–null directions and null geodesics in null directions. While the latter are determined by their initial direction up to a projective family of reparametrizations, there is additional freedom for conformal circles.

Now the results of Lemma 3.4 can be converted to geometry.

Proposition 3.5. *Let $(M^n, [g])$ be a smooth manifold endowed with a pseudo–Riemannian conformal structure of signature (p, q), where $n = p + q \geq 3$. Suppose η is a conformal Killing vector field with a higher order fixed point at $x_0 \in M$ with non–null isotropy $\alpha \in T^*_{x_0}M$.*

(1) *The point x_0 is smoothly isolated in the strongly fixed set. Null geodesics emanating from x_0 in directions in* $\ker(\alpha)$ *are, locally around x_0, fixed points of η.*
(2) *Let $\xi_0 \in T_{x_0}M$ be the unique vector such that $g_x(\xi_0, -)$ is a non–zero multiple of α and such that $\alpha(\xi_0) = 2$. Then for any conformal circle $c = c(s)$ in M with $c(0) = x_0$ and $c'(x_0) = \xi_0$, there is $\epsilon > 0$ such that $\varphi^t_\eta(c(s)) = c(\frac{s}{1+st})$ for $|s| < \epsilon$ and $st > 0$. Finally, an open neighborhood of $\{c(s) : |s| < \epsilon\}$, which in particular contains $x_0 = c(0)$ in its closure, is conformally flat.*

Proof: Varying the point $b_0 \in p^{-1}(x_0)$ gives all normal coordinate charts centered at x_0, which give all conformal circles emanating from x_0. Now the result follows from Propositions 2.1 and 2.2, which apply in our case by Lemma 3.4. $\diamondsuit$

3.2.2. *Null isotropy.* Again we start with an algebraic lemma:

Lemma 3.6. *Let $Z \in \mathfrak{g}_1$ be isotropic. Then the sets associated to Z are*

$$\mathbf{R} \cdot \mathbb{I}Z^t = C(Z) \subset F(Z) = \{X \in \mathfrak{g}_{-1} : ZX = \langle X, X\rangle = 0\}$$
$$T(Z) = \{X \in \mathfrak{g}_{-1} : ZX = 1, \langle X, X\rangle = 0\},$$

and all conditions of Proposition 2.2 are satisfied by Z.

Proof: As in the proof of Lemma 3.4, $[Z, X] = 0$ implies $ZX = 0$, and then $[[Z, X], Y] = 0$ implies that X and $\mathbb{I}Z^t$ must be linearly dependent. Conversely, $[Z, \mathbb{I}Z^t] = 0$ is easily verified, so the claim on $C(Z)$ follows. If X is linearly independent of $\mathbb{I}Z^t$, then $0 = [[Z, X], X] = -2XZX + \langle X, X\rangle\mathbb{I}Z^t$ implies $ZX = \langle X, X\rangle = 0$ and thus the description of $F(Z)$.

Since $\langle Z, Z\rangle = 0$, the equation $[[Z, X], Z] = 2Z$ is equivalent to $ZX = 1$. In this case, $[[Z, X], X] = 2X$ is equivalent to $\langle X, X\rangle = 0$, and the description of $T(Z)$ follows. Taking $X \in T(Z)$, the eigenspace decomposition of $\mathfrak{g}_{-1}$ with respect to $A = [Z, X]$ follows from the formula for $[[Z, X], Y]$: The isotropic lines spanned by X and by $\mathbb{I}Z^t$ are the eigenspaces for eigenvalues -2 and 0, respectively, and A acts by multiplication by -1 on the complementary subspace $\ker(Z) \cap X^{\perp}$. In particular, condition (1) from Proposition 2.2 is satisfied.

The grading on $\mathfrak{g}_{-1}$ is the one on $\mathbf{R}^{p+1,q+1}$ that gives rise to the initial $|1|$–grading on $\mathfrak{g}$, shifted by one degree. This shift does not change the induced grading of $\mathfrak{so}(\mathfrak{g}_{-1})$, which thus has eigenvalues -1, 0, and 1. On the other hand, the eigenvalues on $\Lambda^2\mathfrak{g}_1$ are 1, 2, and 3. Hence for any subrepresentation $\mathbb{W}$ of $\Lambda^2\mathfrak{g}_1 \otimes \mathfrak{g}_1$, we have $\mathbb{W}_{st}(A) = 0$, while for a subrepresentation $\mathbb{W}$ of $\Lambda^2\mathfrak{g}_1 \otimes \mathfrak{so}(\mathfrak{g}_{-1})$, we have $\mathbb{W}_{ss}(A) = 0$, and $\mathbb{W}_{st}(A)$ is contained in the tensor product of the degree one part of $\Lambda^2\mathfrak{g}_1$ with the degree -1 part of $\mathfrak{so}(\mathfrak{g}_{-1})$. This implies that all values in $\mathfrak{so}(\mathfrak{g}_{-1})$ of any element of $\mathbb{W}_{st}(A)$ act trivially on the (-2)–eigenspace in $\mathfrak{g}_{-1}$ and thus on X. But $\mathfrak{g}_{-1}$ is spanned by elements of $T(Z)$, so $\cap_{X\in T(Z)}\mathbb{W}_{st}(A) = 0$ follows. $\diamondsuit$

Now we can directly apply Propositions 2.1 and 2.2 to obtain:

Proposition 3.7. *Let $(M^n, [g])$ be a conformal manifold of signature (p, q), where $n = p + q \geq 3$. Suppose that η is a conformal Killing field with a higher order fixed point at $x_0 \in M$ with null isotropy $\alpha \in T^*_{x_0}M$. Let $F(\alpha) \subset T_{x_0}M$ be the set of those null vectors ξ which satisfy $\alpha(\xi) = 0$, and let $C(\alpha) \subset F(\alpha)$ be the line of all elements dual to a multiple of α.*

Then for any normal coordinate chart centered at x_0 with values in $T_{x_0}M$, there is an open neighborhood U of zero in $T_{x_0}M$ such that:

1. *Elements of $F(\alpha) \cap U$ are fixed points of η and $C(\alpha) \cap U$ is contained in the strongly fixed component of x_0.*
2. *For any null vector $\xi \in U$ with $\alpha(\xi) \geq 0$, we get $\varphi^t_\eta(\xi) = \frac{1}{1+t\alpha(\xi)}\xi$.*

(3) *Any null vector* $\xi \in U$ *with* $\alpha(\xi) > 0$ *has an open neighborhood on which the geometry is flat. In particular, the geometry is flat on an open set containing* x_0 *in its closure.*

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