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CORRIGENDA TO "CHARACTERIZATIONS OF MONADIC NIP"

SAMUEL BRAUNFELD AND MICHAEL C. LASKOWSKI

Abstract. The authors correct results in "Characterizations of monadic NIP" [Trans. Amer. Math. Soc. Ser. B 8 (2021), pp. 948–970]. The notion of endless indiscernible triviality is introduced and replaces indiscernible triviality throughout, in particular in Theorem 1.1. The claim regarding the failure of 4-wqo in Theorem 1.2 is withdrawn and remains unproved.

1. Indiscernible triviality

In Theorem 1.1 of [1], several equivalents of a theory being monadically NIP are given. With the definition of indiscernible-triviality given there, (6) is not equivalent, as can be seen by Example 1.3. However, by making a slight variation on the definition of indiscernible-triviality the equivalence of (6) with the other properties is maintained. Call a linear order $(I,<)$ *endless* if it has neither a minimum nor a maximum element. Clearly, any endless linear order is infinite.

Definition 1.1. A theory T has *endless indiscernible triviality* if for every **endless** indiscernible sequence $\mathcal{I} = (\bar{a}_i : i \in I)$ and every set B of parameters, if $\mathcal{I}$ is indiscernible over each $b \in B$ then $\mathcal{I}$ is indiscernible over B.

This is the same as the definition of indiscernible-triviality, except that *infinite* has been replaced by *endless*.

With this note, we prove Theorem 1.2.

Theorem 1.2. *Replacing indiscernible-triviality by endless indiscernible triviality, the six statements described in* [1, Theorem 1.1] *are equivalent.*

Before launching into the proof of Theorem 1.2, we highlight what the problem was in [1]. The first issue is the Furthermore clause in [1, Lemma 2.18], used in the proof of [1, Proposition 3.11]. We thank James Hanson for providing a counterexample to this clause. The second issue is that in the proof of [1, Proposition 3.11], we assumed that the failure of indiscernible triviality could be witnessed by a $\mathbb{Q}$-indexed sequence, obliterating the distinction between indiscernible triviality and endless indiscernible triviality. To see that (full) indiscernible triviality can fail in a monadically NIP theory, we thank Artem Chernikov for indicating Example 1.3.

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Example 1.3. Let T_{dt} be the theory of dense meet-trees as in [3, Section 2.3.1]. By [2, Corollary 2.8], T_{dt} is monadically NIP. (It is also fairly easy to check the quantifier-free type-counting criterion in [1, Proposition 4.8] over indiscernible sequences of singletons, which is sufficient.) Let $M \models T_{dt}$, let $\mathcal{I} = (a_i : i \in \omega)$ be a decreasing sequence, and let $b, b' \in M$ be such that $b, b' > a_0$ and $b \wedge b' = a_0$. Then $\mathcal{I}$ is indiscernible over b and over b', but not over bb'.

In the remainder of this section, we prove Theorem 1.2 and indicate where the endlessness assumption is used. Definition 1.4, which appears in [3], is standard.

Definition 1.4. Two sequences $(\bar{a}_i : i \in I)$ and $(\bar{b}_j : j \in J)$ (not necessarily of the same arities) are *mutually indiscernible over* C if $(\bar{a}_i : i \in I)$ is indiscernible over $C \cup \bigcup\{\bar{b}_j : j \in J\}$ and $(\bar{b}_j : j \in J)$ is indiscernible over $C \cup \bigcup\{\bar{a}_i : i \in I\}$.

In [1], in order to recover Theorem 1.1, it suffices to recover Proposition 3.11, so in the notation there, define

(2*) T is dp-minimal and has endless indiscernible triviality

which is identical to the existing (2), but now with endless indiscernible triviality replacing indiscernible-triviality.

Again in the notation of Proposition 3.11, we must show that (2*) $\Rightarrow$ (3) and that (1) $\Rightarrow$ (2*).

The existing proof that (2) $\Rightarrow$ (3) is easily modified to show (2*) $\Rightarrow$ (3). The only issue is that the convex piece I' containing $\bar{a}_i$ might not be endless. But in this case, the convex piece containing $\bar{a}_j$ must be $I \backslash I'$, which is endless. So we may conclude the argument substituting $\bar{a}_j$ for $\bar{a}_i$ and $I \backslash I'$ for I'.

Establishing the implication (1) $\Rightarrow$ (2*) is more involved, where (1) states that T has the f.s. dichotomy. Without going through the problematic (2), the paper still contains a proof of (1) $\Rightarrow$ (4), where (4) asserts that there T does not admit a precoding configuration. Before tracing this proof, we recall these definitions from [1].

Definition 1.5. T has the *f.s. dichotomy* if, for all models M, all finite tuples $\bar{a}, \bar{b} \in \mathfrak{C}$, if $\operatorname{tp}(\bar{b}/M\bar{a})$ is finitely satisfied in M, then for any singleton c, either $\operatorname{tp}(\bar{b}/M\bar{a}c)$ or $\operatorname{tp}(\bar{b}c/M\bar{a})$ is finitely satisfied in M.

Definition 1.6. A *precoding configuration* consists of a $\phi(\bar{x}, \bar{y}, z)$ with parameters and a sequence $\mathcal{I} = \langle \bar{d}_i : i \in \mathbb{Q} \rangle$, indiscernible over the parameters of ϕ, such that for some (equivalently, for every) $s < t$ from $\mathbb{Q}$, there is $c \in \mathfrak{C}$ such that

(1) $\mathfrak{C} \models \phi(\bar{d}_s, \bar{d}_t, c)$;
(2) $\mathfrak{C} \models \neg\phi(\bar{d}_s, \bar{d}_v, c)$ for all $v > t$; and
(3) $\mathfrak{C} \models \neg\phi(\bar{d}_u, \bar{d}_t, c)$ for all $u < s$.

In [1, §4], it is proved that if T has the f.s. dichotomy, then T does not admit coding on tuples, which is condition (3) in [1, Proposition 3.11]. Thus the implication (3) $\Rightarrow$ (4) in [1, Proposition 3.11] shows that if T has the f.s. dichotomy then T does not admit a precoding configuration. (We take this opportunity to note that after the first sentence in the proof of (3) $\Rightarrow$ (4) in [1, Proposition 3.11], the following should be inserted: "By Ramsey and compactness, we may assume that the truth value of $\phi(\bar{a}_i, \bar{a}_j, c_{k,\ell})$ depends only on the order-type of $ijk\ell$.")

Evidently, the existence of a precoding configuration is a statement about a certain configuration being consistent with T, hence one can use compactness to construct such configurations from many variations. We record two variants in Lemma 1.7.

Lemma 1.7. *T admits a precoding configuration if either of the following hold:*

(1) *There is a sequence $(\bar{d}_i : i \in \mathbb{Z})$ (not necessarily indiscernible) and a formula $\phi(\bar{x}, \bar{y}, z)$ such that, for **every** $s < 0 < t$ there is $h_{s,t} \in \mathfrak{C}$ such that*
- $\models \phi(\bar{d}_s, \bar{d}_t, h_{s,t})$;
- $\models \neg\phi(\bar{d}_u, \bar{d}_t, h_{s,t})$ *for every* $u < s$; *and*
- $\models \neg\phi(\bar{d}_s, \bar{d}_v, h_{s,,t})$ *for every* $v > t$.

(2) *Or there is an indiscernible sequence $(\bar{d}_i : i \in \mathbb{Z})$ and a formula $\phi(\bar{x}, \bar{y}, z)$ such that, for **some** $s < 0 < t$ there is $h_{s,t} \in \mathfrak{C}$ such that*
- $\models \phi(\bar{d}_s, \bar{d}_t, h_{s,t})$;
- $\models \neg\phi(\bar{d}_u, \bar{d}_t, h_{s,t})$ *for every* $u < s$; *and*
- $\models \neg\phi(\bar{d}_s, \bar{d}_v, h_{s,t})$ *for every* $v > t$.

Proof. (1) is immediate by compactness. For (2), we first extend our given indiscernible sequence $(\bar{d}_i : i \in \mathbb{Z})$ to an indiscernible sequence $(\bar{d}_i : i \in \mathbb{Q})$, maintaining the extra conditions that $\neg\phi(\bar{d}_i, \bar{d}_t, h_{s,t})$ for all $i < s$, $i \in \mathbb{Q}$ and that $\neg\phi(\bar{d}_s, \bar{d}_i, h_{s,t})$ for all $i > t$, $i \in \mathbb{Q}$ in three steps, all using compactness. First, since $(\bar{d}_i : i < s, i \in \mathbb{Z})$ is an infinite, indiscernible sequence over $(\bar{d}_i : i \geq s)$, for which $\neg\phi(\bar{d}_i, \bar{d}_t, h_{s,t})$ for every such i, by compactness there is an extension of this segment to $(\bar{d}_i : i < s, i \in \mathbb{Q})$ maintaining indiscernibility of the entire expanded sequence, as well as $\neg\phi(\bar{d}_i, \bar{d}_t, h_{s,t})$. Dually, we can find an extension $(\bar{d}_i : i > t, i \in \mathbb{Q})$ of $(\bar{d}_i : i > t, i \in \mathbb{Z})$ maintaining indiscernibility with $\neg\phi(\bar{d}_s, \bar{d}_i, h_{s,t})$ for every $i > t$, $i \in \mathbb{Q}$. Finally, for the middle segment $(\bar{d}_i : s < i < t, i \in \mathbb{Z})$, we only need to maintain indiscernibility. Although the sequence $(d_i : s < i < t, i \in \mathbb{Z})$ is finite, it is part of an endless indiscernible sequence. Thus, it follows by compactness that there is an extension $(\bar{d}_i : s < i < t, i \in \mathbb{Q})$ of $(\bar{d}_i : s < i < t, i \in \mathbb{Z})$, for which the entire sequence $(\bar{d}_i : i \in \mathbb{Q})$ is indiscernible. So we have constructed an indiscernible sequence $(\bar{d}_i : i \in \mathbb{Q})$ with some distinguished pair $s < t$ for which a witnessing element $h_{s,t}$ exists. However, as $Aut(\mathbb{Q}, <)$ is 2-homogeneous and since every $\sigma \in Aut(\mathbb{Q}, <)$ induces an automorphism of $\mathfrak{C}$, we conclude that for every $s' < t'$, a witnessing element $h_{s',t'}$ exists. Thus, we obtain a precoding configuration. □

We now assume T has the f.s. dichotomy. The proof that T is dp-minimal in the existing proof of (1) ⇒ (2) in [1, Proposition 3.11] is unchanged. In fact, the proof gives the following stronger statement.

Lemma 1.8. *If T has the f.s. dichotomy, then for every indiscernible sequence $\mathcal{I} = (\bar{a}_i : i \in I)$ and singleton b, there is a partition $I = I_0^\frown I_1^\frown I_2$ where I_1 is either empty or a singleton, such that $(\bar{a}_i : i \in I_0)$ and $(\bar{a}_i : i \in I_2)$ are mutually indiscernible over $b\mathcal{I}_1$.*

In the case where I is Dedekind complete, we may assume I_1 is a singleton.

We will assume that T has the f.s. dichotomy but fails endless indiscernible triviality and eventually arrive at one of the two clauses of Lemma 1.7, giving our contradiction. Endlessness of $(I,<)$ is crucial as once we cut the indiscernible sequence $\mathcal{I}$ into two mutually indiscernible halves, we still have that each half is an infinite indiscernible sequence and thus can be extended. In a nutshell, this extendibility of each half is what is failing in Example 1.3.

Lemma 1.9. *Suppose T has the f.s. dichotomy, $(I,<)$ is an endless, Dedekind complete linear order, $\mathcal{I} = (\bar{a}_i : i \in I)$ is indiscernible over $\emptyset$, but not over b for some singleton $b \in \mathfrak{C}$. Then there are $i^* \in I$, a finite $F \subseteq \mathfrak{C}$, and an F-definable $\delta(\bar{x}, y)$ such that*

(1) *$(\bar{a}_i : i \in I)$ is indiscernible over F;*
(2) *the subsequences $(\bar{a}_i : i < i^*)$ and $(\bar{a}_i : i > i^*)$ are mutually indiscernible over $Fb\bar{a}_{i^*}$; and*
(3) *the sequence of truth values of $(\delta(\bar{a}_i, b) : i \in I)$ is not constant.*

Proof. Since $\mathcal{I}$ is not indiscernible over b, we apply Lemma 1.8, and let i^* be the singleton element of I_1 there. Since $(I,<)$ is endless, choose $i^*_-, i^*_+ \in I$ with $i^*_- < i^* < i^*_+$. Choose a formula $\phi(\bar{x}_1, \ldots, \bar{x}_n, b)$ witnessing that $\mathcal{I}$ is not indiscernible over b. By mutual indiscernibility over $\bar{a}_{i^*}b$, there must be some $1 \le k \le n$ for which: for some/every $i_1 < i_2 < \cdots < i_{k-1} < i^*_-$, for some/every $i^*_+ < i_{k+1} < \cdots < i_n$, the truth values of these three statements are nonconstant:

- $\phi(\bar{a}_{i_1}, \ldots, \bar{a}_{i_{k-1}}, \bar{a}_{i^*_-}, \bar{a}_{i_{k+1}}, \ldots, \bar{a}_{i_n}, b)$;
- $\phi(\bar{a}_{i_1}, \ldots, \bar{a}_{i_{k-1}}, \bar{a}_{i^*}, \bar{a}_{i_{k+1}}, \ldots, \bar{a}_{i_n}, b)$; and
- $\phi(\bar{a}_{i_1}, \ldots, \bar{a}_{i_{k-1}}, \bar{a}_{i^*_+}, \bar{a}_{i_{k+1}}, \ldots, \bar{a}_{i_n}, b)$.

Let $I' := (\ell_1, \ldots, \ell_{k-1}) \frown I \frown (r_{k+1}, \ldots, r_n)$ extend I. By compactness, choose $n-1$ new tuples $(\bar{a}_{\ell_1}, \ldots, \bar{a}_{\ell_{k-1}})$, $(\bar{a}_{r_{k+1}}, \ldots, \bar{a}_{r_n})$ such that the extended sequences $(\bar{a}_i : i < i^*, i \in I')$ and $(\bar{a}_i : i > i^*, i \in I')$ remain mutually indiscernible over $\bar{a}_{i^*}b$. Put

$$F := \bigcup\{\bar{a}_{\ell_i} : 1 \le i \le k-1\} \cup \bigcup\{\bar{a}_{r_i} : k+1 \le i \le n\},$$

and let $\delta(\bar{x}, y) := \phi(\bar{a}_{\ell_1}, \ldots, \bar{a}_{\ell_{k-1}}, \bar{x}, \bar{a}_{r_{k+1}}, \ldots, \bar{a}_{r_n}, y)$. This works. □

Lemma 1.10. *Suppose T has the f.s. dichotomy. Then T has endless indiscernible triviality.*

Proof. Assume, by way of contradiction, that T fails endless indiscernible triviality. An easy induction on $|B|$ gives that there is some finite A and singletons b, c for which some endless $(I,<)$ supports a sequence $(\bar{a}_i : i \in I)$ that is indiscernible over Ab and Ac, but not over Abc. By adding constants to the language we may assume $A = \emptyset$ and, as $(\mathbb{Z},<)$ embeds into any endless linear order, we may take $I = \mathbb{Z}$. Summarizing, we assume the existence of a sequence $(\bar{a}_i : i \in \mathbb{Z})$ that is indiscernible over b and c individually, but not over bc. Now, working over c, apply Lemma 1.9 to this sequence and b to obtain $i^* \in \mathbb{Z}$, a finite set F and an Fc-definable $\delta(\bar{x}, b)$ as there. To make the dependence on c explicit, write δ as $\delta(\bar{x}, y, c)$, so $\delta(\bar{x}, y, z)$ is F-definable. As $(\mathbb{Z},<)$ is transitive, we may assume $i^* = 0$. We summarize the situation from the point of view of b, which we now label as b_0.

We have the following:

(1) $(\bar{a}_i : i \in \mathbb{Z})$ is indiscernible over Fc.
(2) For $b = b_0$,
 (a) $(\bar{a}_i : i \in \mathbb{Z})$ is indiscernible over Fb_0;
 (b) $(\bar{a}_i : i < 0)$ and $(\bar{a}_i : i > 0)$ are mutually indiscernible over Fcb_0; and
 (c) the truth value of $(\delta(\bar{a}_i, , b_0, c) : i \in \mathbb{Z})$ is nonconstant.

Because of (1), there is an automorphism σ of $\mathfrak{C}$ fixing Fc with $\sigma(\bar{a}_i) = \sigma(\bar{a}_{i+1})$ for all $i \in Z$. Let $b_j := \sigma^j(b)$, the j-fold iteration of σ (this also makes sense for $j = 0$ and $j < 0$). Thus, with the same Fc and $(\bar{a}_i : i \in \mathbb{Z})$, we have that for every $j \in \mathbb{Z}$, $(\bar{a}_i : i \in Z)$ is indiscernible over Fb_j: $(\bar{a}_i : i < j)$ and $(\bar{a}_i : i > j)$ are mutually indiscernible over Fcb_j; and the truth value of $(\delta(\bar{a}_i, b_j, c) : i \in \mathbb{Z})$ is nonconstant. We remark that we have again made crucial use of the endlessness of our indiscernible sequence to extend b from a singleton to a whole sequence.

Now, keeping Fc fixed, we 'couple' each b_j by its corresponding $\bar{a}_j$, and then by Ramsey's Theorem and compactness we get that for any endless $(J, <)$ there are tuples $(\bar{a}_j b_j : j \in J)$ (possibly distinct from the original elements) satisfying the following conditions:

(1) The sequence $((\bar{a}_j b_j) : j \in J)$ is indiscernible over Fc.
(2) For all $j \in J$,
 (a) the sequence $(\bar{a}_i : i \in J)$ is indiscernible over Fb_j;
 (b) the subsequences $(\bar{a}_i : i < j)$ and $(\bar{a}_i : i > j)$ are mutually indiscernible over Fcb_j; and
 (c) the truth values of $(\delta(\bar{a}_i, , b_j, c) : i \in J)$ is nonconstant.

Claim 1 will allow us to define a precoding configuration.

Claim 1. There is a sequence $(\bar{d}_r b_r : r \in \mathbb{R})$ that is indiscernible over Fc with $(\bar{d}_r : r \in \mathbb{R})$ indiscernible over Fb_0 and an F-definable formula $\psi(\bar{x}, y, z)$ such that

(1) for all $r, s \in \mathbb{R}$, $\models \psi(\bar{d}_r, b_s, c)$ if and only if $r = s$; and
(2) for every singleton $c' \in \mathfrak{C}$ and $r \in \mathbb{R}$, there is at most one $s \in \mathbb{R}$ such that $\models \psi(\bar{d}_s, b_r, c')$.

Proof of Claim 1. Consider the sequence $(\bar{a}_j b_j : j \in J)$ obtained above with $J = 3 \times \mathbb{R}$. As notation, for each $r \in \mathbb{R}$ write each 'triple' as $(\bar{a}_{r_-}, \bar{a}_r, \bar{a}_{r_+})$ and let $\bar{d}_r := \bar{a}_{r_-}\bar{a}_r\bar{a}_{r_+}$ be the concatenation of the triple. In what follows we only consider b_r for each $r \in \mathbb{R}$. Finally, put

$$\psi(\bar{x}_-\bar{x}\bar{x}_+, y, z) := \neg\left[\delta(\bar{x}_-, y, z) \leftrightarrow \delta(\bar{x}, y, z) \leftrightarrow \delta(\bar{x}_+, y, z)\right].$$

Thus, we have $(\bar{d}_r b_r : r \in \mathbb{R})$ is indiscernible over Fc, and for each $r \in \mathbb{R}$ we have $(\bar{d}_s : s \in \mathbb{R})$ is indiscernible over Fb_r and the pair of subsequences $(\bar{d}_s : s < r)$ and $(\bar{d}_s : s > r)$ are mutually indiscernible over Fcb_r. Moreover, for any $r, s \in \mathbb{R}$, $\models \psi(\bar{d}_s, b_r, c)$ if and only if $r = s$.

To get the final clause, choose any $c' \in \mathfrak{C}$ and $r \in \mathbb{R}$. We know the original sequence $(\bar{a}_i : i \in 3 \times \mathbb{R})$ is indiscernible over Fb_r. If it is also indiscernible over $Fb_r c'$, then the truth value of $(\delta(\bar{a}_i, b_r, c') : i \in 3 \times \mathbb{R})$ is constant, hence $\models \neg\psi(\bar{d}_s, b_r, c')$ holds for every $s \in \mathbb{R}$. On the other hand, if it fails to be indiscernible over $Fb_r c'$, then, working over Fb_r, we apply Lemma 1.9 to the sequence $(\bar{a}_i : i \in 3 \times \mathbb{R})$. Choose $i^* \in 3 \times \mathbb{R}$ for which the subsequences $(\bar{a}_i : i < i^*)$ and $(\bar{a}_i : i > i^*)$ are mutually indiscernible over $Fb_r c'$. Choose $s \in \mathbb{R}$ such that $i^* \in \{s_-, s, s_+\}$. Then for any $t \neq s$, with $t \in \mathbb{R}$, the triple (t_-, t, t_+) lies in one of the two subsequences. Thus, by indiscernibility we have $\models \neg\psi(\bar{d}_t, b_r, c')$ for all $t \neq s$. $\Diamond$

Continuing, as $(\bar{d}_i : i \in \mathbb{R})$ is indiscernible over b_0, choose an automorphism $\sigma \in Aut(\mathfrak{C})$ such that $\sigma(\bar{d}_j) = \bar{d}_{j+1}$ for every $j \in \mathbb{R}$, and also $\sigma(b_0) = b_0$. For each $i \in \mathbb{Z}^+$, let $\sigma^{(i)}$ denote the i-fold composition of σ, so e.g., $\sigma^{(i)}(\bar{a}_j) = \bar{a}_{j+i}$, while $\sigma^{(i)}(b_0) = b_0$. As notation, put $c_i := \sigma^{(i)}(c)$.

For each $m \in \mathbb{Z}^+$, let $B_m = \{\, j \in (-\infty, 0) \mid\, \models \psi(\bar{d}_j, b_j, c_m) \,\}$. There are now two cases, each of which leads to a precoding configuration.

Case 1. Some B_m is not well ordered.

Fix such an $m \in \mathbb{Z}^+$. Fix a strictly decreasing sequence $J = (j_n : n \in \omega)$ from B_m and put $I := (i \in \mathbb{Z}^+ : i \geq m)$. Thus $K := J^\frown 0^\frown I$ describes a subordering of $(\mathbb{R}, <)$ of order type $(\mathbb{Z}, <)$. For $k \in K$, let $\bar{e}_k$ denote the concatenation $\bar{d}_k b_k$ and let $\theta(\bar{x}_1 y_1, \bar{x}_2 y_2, z) := \psi(\bar{x}_2, y_1, z)$. That $(\bar{e}_k : k \in K)$ and θ satisfy the hypotheses of Lemma 1.7(2) with $s = 0$, $t = m$, and $h_{s,t} = c_m$ follows from Claim 2.

Claim 2.

(1) $\models \psi(\bar{d}_m, b_0, c_m)$;
(2) $\models \neg\psi(\bar{d}_i, b_0, c_m)$ for all $i > m$; and
(3) $\models \neg\psi(\bar{d}_m, b_j, c_m)$ for all $j \in J$.

Proof of Claim 2. For (1), we have $\models \psi(\bar{d}_0, b_0, c)$, hence applying σ_m gives $\models \psi(\sigma_m(\bar{d}_0), \sigma_m(b_0), \sigma_m(c))$, i.e., $\models \psi(\bar{d}_m, b_0, c_m)$.

For (2), we know that for any $k > 0$, $\models \neg\psi(\bar{d}_k, b_0, c)$, so applying σ_m yields $\models \neg\psi(\bar{d}_{k+m}, b_0, c_m)$.

For (3), since $j \in B_m$, we have $\models \psi(\bar{d}_j, b_j, c_m)$. But then by the final clause of Claim 1, we have $\models \neg\psi(\bar{d}_m, b_j, c_m)$ for $m \neq j$. $\Diamond$

Case 2. Not Case 1, i.e., every B_m is well ordered.

In this case, for any $i \in \mathbb{Z}^+$, the shifted set

$$B_m + i = \{\, r \in (-\infty, 0) \mid r = b + i \text{ for some } b \in B_m, i \in \mathbb{Z}^+ \,\}$$

is well ordered as well. Since any well-ordered subset of $(-\infty, 0)$ is nowhere dense, it follows by Baire category that the complement

$$S = \{\, r \in (-\infty, 0) \mid r \notin B_m + i \text{ for every } i, m \in \mathbb{Z}^+ \,\}$$

is not nowhere dense. Thus, S contains a strictly decreasing sequence $J = (j_n : n \in \omega)$, so $K := J^\frown 0^\frown \mathbb{Z}^+$ has order type $(\mathbb{Z}, <)$. For each $k \in K$, let $\bar{e}_k$ denote the concatenation $\bar{d}_k c_k$, let $b_{i,j} := \sigma^{(i)}(b_{j-i})$, and let $\theta(\bar{x}_1 z_1, \bar{x}_2 z_2, y) := \psi(\bar{x}_1, y, z_2)$. Here, we will get an instance of precoding via Lemma 1.7(1), as witnessed by $(\bar{e}_k : k \in K)$, θ, and the witnesses $b_{i,j}$ for $j < 0 < i$ from K, once we establish Claim 3.

Claim 3. For every $j \in S$ and $i \in \mathbb{Z}^+$,

(1) $\models \psi(\bar{d}_j, b_{i,j}, c_i)$;
(2) for all $j' \in S \setminus \{ j \}$, $\models \neg\psi(\bar{d}_{j'}, b_{i,j}, c_i)$; and
(3) for all $\ell > i$, $\models \neg\psi(\bar{d}_j, b_{i,j}, c_\ell)$.

Proof of Claim 3. For (1) and (2): By Claim 1(1), we have $\models \psi(\bar{d}_{j-i}, b_{j-i}, c)$ and for any $t \neq j - i$ we have $\models \neg\psi(\bar{d}_t, b_{j-i}, c)$. Applying $\sigma^{(i)}$ yields $\models \psi(\bar{d}_j, b_{i,j}, c_i)$, but $\models \neg\psi(\bar{d}_{j'}, b_{i,j}, c_i)$ for any $j' \neq j$.

For (3), given $\ell > i$, put $m := \ell - i$. Since $j \in S$, $j - i \notin B_m$, so $\models \neg\psi(\bar{d}_{j-i}, b_{j-i}, c_m)$. Then, applying $\sigma^{(i)}$ (and using $c_\ell = \sigma^{(i)}(c_m)$) yields $\models \neg\psi(\bar{d}_j, b_{i,j}, c_\ell)$, as required. ♢

□

2. Well-quasi-order

In [1, Theorem 5.3], the proof that $Age(T)$ is not 4-wqo is flawed. The issue is that the formula $\phi^*(x, y, z)$ is not existential, and thus neither is the formula $\exists z \phi^*(x, y, z)$ that we use to define the edges of our graphs. Since the formula is not existential, it need not be preserved by embeddings. Thus the first sentence of the last paragraph of the proof is unjustified.

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Computer Science Institute, Charles University, Prague 18000, Czechia; and Computer Science Institute, Czech Academy of Science, Prague 18200, Czechia

Department of Mathematics, University of Maryland, College Park, MD 20742
Email address: mcl@math.umd.edu