

# EXISTENTIAL CHARACTERIZATIONS OF MONADIC NIP

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ABSTRACT. We show that if a universal theory is not monadically NIP, then this is witnessed by a canonical configuration defined by an existential formula. As a consequence, we show that a hereditary class of relational structures is NIP (resp. stable) if and only if it is monadically NIP (resp. monadically stable). As another consequence, we show that if such a class is not monadically NIP, then it has superexponential growth rate.

## 1. INTRODUCTION

When studying the combinatorial behavior of the models of a complete theory, the model-theoretic properties of stability and NIP repeatedly appear as dividing lines between tame and wild behavior. More recently this has been shown to be true when studying the combinatorics of hereditary classes of relational structures, i.e. classes closed under taking substructure. For example, in the graph sparsity theory initiated by Nešetřil and Ossona de Mendez, the fundamental dichotomy for monotone graph classes (Definition 4.8) is nowhere dense/somewhere dense, and [1] showed that nowhere denseness agrees with (monadic) stability and (monadic) NIP for such classes. For hereditary classes of ordered graphs, [4] showed that bounded twin-width yields a fundamental dichotomy, and that it agrees with (monadic) NIP.

As seen in these examples, monadically stable/NIP theories, i.e. theories that remain stable/NIP after arbitrary expansions by unary predicates, also naturally appear for hereditary classes. These theories were first studied in [2, 15] and more recently by the authors in [5], which showed these properties give significant structural information beyond their non-monadic counterparts.

But why do these monadic properties appear as dividing lines in hereditary classes? Their failure only guarantees that we can produce wild behavior after expanding by unary predicates, but many of the problems where they appear do not involve such expansions. Generalizing results mentioned above concerning the collapse of stability/NIP to their monadic variants,

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we show passing to substructures suffices to produce the wild behavior obtained by monadic expansions. The monadic versions of properties appear as dividing lines in hereditary classes because there they agree with their non-monadic counterparts.

**Theorem 1.1** (Theorem 4.6, Theorem 4.9). *Let  $\mathcal{P} \in \{ \text{stable}, \text{NIP} \}$ . Let  $\mathcal{C}$  be a hereditary class of relational structures. The following are equivalent.*

- (1)  $\text{Th}(\mathcal{C})_{\forall}$  is monadically  $\mathcal{P}$ .
- (2)  $\text{Th}(\mathcal{C})$  is monadically  $\mathcal{P}$ .
- (3)  $\text{Th}(\mathcal{C})_{\forall}$  is  $\mathcal{P}$ .
- (4)  $\text{Th}(\mathcal{C})$  is  $\mathcal{P}$ .

*Furthermore, if  $\mathcal{C}$  is monotone, then  $\text{Th}(\mathcal{C})$  is NIP if and only if  $\text{Th}(\mathcal{C})$  is monadically stable.*

Since the models of a monadically NIP theory seemingly have a tree-like structure (see the characterization in [3], while the characterizations in [5] even show an order-like structure if we look coarsely enough), this theorem suggests that we should expect a sharp dichotomy in hereditary classes: either the structures admit tree-like decompositions, or the class is at least as complicated as the class of all graphs. This dichotomy may also partially explain the ubiquity of tree-like decompositions in structural graph theory.

The key step behind Theorem 1.1 is that if a universal theory is not monadically NIP, then this is witnessed by a configuration defined by an *existential* formula.

**Theorem 1.2** (Theorem 3.17). *For a universal theory  $T$ , the following are equivalent.*

- (1)  $T$  is monadically NIP.
- (2)  $T$  does not admit pre-coding by an existential formula.
- (3)  $T$  has the e-f.s. dichotomy

*In particular, if  $T$  is not monadically NIP then it admits pre-coding by an existential formula.*

While [5] showed a pre-coding configuration must definably appear in theories that are not monadically NIP, the existential condition allows for much greater control and more direct finitization, and we expect further applications of this result, as in [6].

To illustrate that Theorem 1.2 makes (the failure of) monadic NIP manifest within standardly-considered combinatorics, we finish by using it to show that if a hereditary class  $\mathcal{C}$  is not monadically NIP then it has super-exponential growth rate.

**Theorem 1.3** (Theorem 5.3). *Let  $\mathcal{C}$  be a hereditary class of relational structures. If  $\text{Th}(\mathcal{C})$  is not monadically NIP, then there is some  $k \in \omega$  such that the unlabeled growth rate  $f_{\mathcal{C}}(n) = \Omega(\lfloor n/k \rfloor!)$ .*

After preliminary material in Section 2, Theorem 1.2 is proved in Section 3. The proof recapitulates some of the material of [5] in the setting of

existentially closed models of a universal theory, but must ultimately return to the setting of saturated models and connect to the characterizations given in [5]. Section 3 also shows that if a universal theory is monadically NIP but not monadically stable, then there is an atomic formula witnessing the order property. We prove Theorem 1.1 in Section 4, by manipulating a generalized indiscernible instance of the configuration provided by Theorem 1.2. Further manipulations of this configuration in Section 5 yield Theorem 1.3.

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## 2. PRELIMINARIES

**2.1. Existentially closed models.** Model theory typically takes place in the category whose objects are models of a complete theory  $T$  and whose morphisms are elementary embeddings, although it is also convenient to cast this as working in a large saturated “monster model” of  $T$  containing all other models as elementary submodels. When studying a universal theory  $T$ , it may be more appropriate to work in the category whose objects are existentially closed models of  $T$  and whose morphisms are embeddings. In this section, we review this setting, as well as the analogous notion of “monster model”.

To motivate this switch to universal theories, we begin by showing that for monadic properties, only the universal part of a theory is relevant.

**Definition 2.1.** We will say an incomplete theory  $T$  is (monadically) stable/NIP if this is true for all completions of  $T$ .

**Notation 2.2.** Given a theory  $T$ , we let  $T_{\forall}$  denote the set of universal sentences in  $T$ .

**Fact 2.3.** Let  $T$  be a theory. Then  $M \models T_{\forall}$  if and only if there is some  $N \models T$  such that  $M \subset N$ .

**Lemma 2.4.** Let  $\mathcal{P} \in \{\text{stable}, \text{NIP}\}$ . Suppose  $N$  is monadically  $\mathcal{P}$  and  $M \subset N$ . Then  $M$  is monadically  $\mathcal{P}$ .

*Proof.* If  $M$  is not monadically  $\mathcal{P}$ , then there is some unary expansion  $(M, U_1, \dots, U_k)$  containing arbitrarily large finite approximations to the configuration witnessing the failure of  $\mathcal{P}$ , uniformly defined by  $\phi$ . Let  $U$  be a unary predicate such that  $U(N) = M$ , interpret  $U_i(N) = U_i(M)$  for every  $i \in [k]$ , and let  $\phi'$  be obtained by restricting all quantifiers in  $\phi$  to  $U$ . Then  $\phi'$  also defines arbitrarily large finite approximations to the configuration witnessing  $\mathcal{P}$  in  $(N, U, U_1, \dots, U_k)$ .  $\square$

**Proposition 2.5.** Let  $\mathcal{P} \in \{\text{stable}, \text{NIP}\}$ . Let  $T$  and  $T'$  be theories, such that  $T_{\forall} \subseteq T'_{\forall}$ . If  $T$  is monadically  $\mathcal{P}$ , then  $T'$  is monadically  $\mathcal{P}$ .

In particular,  $T$  is monadically  $\mathcal{P}$  if and only if  $T_{\forall}$  is monadically  $\mathcal{P}$ .

*Proof.* Fix  $M' \models T'$ . Since  $T_{\forall} \subseteq T'_{\forall}$ ,  $M' \models T_{\forall}$ , so by Fact 2.3 there is  $M \models T$  such that  $M' \subset M$ . As  $M$  is monadically  $\mathcal{P}$ , so is  $M'$  by Lemma 2.4.  $\square$

The remainder of this section is largely taken from [14].

**Definition 2.6.** Let  $T$  be a universal theory. Then  $M \models T$  is *existentially closed* if for any  $\bar{m} \in M$  and existential  $\phi(\bar{x})$ , if there is some  $N \supset M$  such that  $N \models T$  and  $N \models \phi(\bar{m})$ , then already  $M \models \phi(\bar{m})$ .

**Definition 2.7.** Let  $T$  be a universal theory and  $\phi(\bar{x})$  be an existential formula. An *obstacle* to  $\phi$  is an existential formula  $\psi(\bar{x})$  such that  $T \models \forall \bar{x}(\psi(\bar{x}) \rightarrow \neg\phi(\bar{x}))$ .

A *maximal existential type* is a maximal consistent (relative to  $T$ ) set of existential formulas. Equivalently, a partial existential type  $p(\bar{x})$  is maximal existential if for every existential formula  $\phi(\bar{x})$ ,  $\phi \notin p$  if and only if there is  $\psi \in p$  that is an obstacle to  $\phi$ .

**Notation 2.8.** Given a structure  $M$  and  $\bar{m} \subset M$ , we will use  $\text{tp}_{\exists}(\bar{m})$  to denote the existential type of  $\bar{m}$ , i.e. the set of existential formulas satisfied by  $\bar{m}$  in  $M$ .

**Fact 2.9.** Let  $T$  be a universal theory. For every  $M \models T$ , there is  $N \supset M$  such that  $N$  is an existentially closed model of  $T$  and  $|N| \leq |M| + |T|$ .

**Fact 2.10.** Let  $T$  be a universal theory. Then  $M \models T$  is existentially closed if and only if for every  $\bar{m} \in M$ ,  $\text{tp}_{\exists}(\bar{m})$  is maximal existential.

Thus existentially closed models of  $T$  can be characterized by omitting certain types, namely those types whose existential part is not maximal existential. Furthermore, the class of existentially closed models satisfies amalgamation with respect to embeddings, leading to our analogue of monster models.

**Definition 2.11.** Let  $T$  be a universal theory and  $M \models T$ . Then  $M$  is an *existentially closed monster model* of  $T$  if it satisfies the following conditions.

- (1) Every partial existential type  $p(\bar{x})$  over  $A \subset M$  with  $|A| < |M|$  that is finitely satisfiable in  $M$  is realized in  $M$ .
- (2) Whenever  $A, B \subset M$ , with  $|A|, |B| < |M|$  and  $f: A \rightarrow B$  is a bijection such that  $\text{tp}_{\exists}(A) \subset \text{tp}_{\exists}(B)$ , then  $f$  extends to an automorphism of  $M$ .

As in the case of monster models of complete theories, the existence of arbitrarily large existentially closed monster models requires set-theoretic assumptions, and this can similarly be avoided by, for example, only requiring the existential versions of saturation and homogeneity for some sufficiently large  $\kappa$ , rather than for  $|M|$ .

If a universal theory  $T$  does not have the joint embedding property, then it will have multiple existentially closed monster models with distinct universal

theories that cannot jointly embed into a single model of  $T$ . So we will sometimes have to quantify over what happens in every existentially closed monster model of  $T$ .

We next introduce the relevant version of elementary substructure.

**Definition 2.12.** Given structures  $M \subset N$ , we write  $M \prec_{\exists} N$  if for every existential formula  $\phi(\bar{x})$  and  $\bar{m} \subset M$ ,  $M \models \phi(\bar{m}) \Leftrightarrow N \models \phi(\bar{m})$ .

Note that if  $N$  is an existentially closed model of  $T$  and  $M \subset N$ , then  $M \prec_{\exists} N$  if and only if  $M$  is also existentially closed.

Finally, as we will make significant use of finite satisfiability, we review some of its properties in the existentially closed setting.

**Fact 2.13.** Let  $M \prec_{\exists} \mathfrak{U}$ .

- (1) Every existential type  $p$  over  $M$  is finitely satisfiable in  $M$ .
- (2) (Non- $e$ -splitting) If  $p$  is an existential type over  $B$  that is finitely satisfied in  $M$ , then  $p$  does not  $e$ -split over  $M$ , i.e., if  $\bar{b}, \bar{b}' \subseteq B$  and  $\text{tp}_{\exists}(\bar{b}/M) = \text{tp}_{\exists}(\bar{b}'/M)$ , then for any  $\phi(\bar{x}, \bar{y})$ , we have  $\phi(\bar{x}, \bar{b}) \in p$  if and only if  $\phi(\bar{x}, \bar{b}') \in p$ .
- (3) (Transitivity) If  $\text{tp}_{\exists}(\bar{b}/C)$  and  $\text{tp}_{\exists}(\bar{a}/\bar{b}C)$  are both finitely satisfied in  $M$ , then so is  $\text{tp}_{\exists}(\bar{a}\bar{b}/C)$ .

*Proof.* (1) As maximal existential types are closed under conjunctions, we may consider a single formula  $\phi(\bar{x}; \bar{m}) \in p$ , with  $\bar{m} \subset M$ . By definition of being existentially closed, there is some  $\bar{n} \in M$  such that  $M \models \phi(\bar{n}; \bar{m})$ .

(2) Suppose  $\phi(\bar{x}, \bar{b}) \in p(\bar{x})$ , but  $\phi(\bar{x}, \bar{b}') \notin p(\bar{x})$ . Then there is  $\psi(\bar{x}, \bar{c}) \in p(\bar{x})$  that is an obstacle for  $\phi(\bar{x}, \bar{b}')$ . By finite satisfiability, there is some  $\bar{m} \in M$  such that  $\mathfrak{U} \models \phi(\bar{m}, \bar{b}) \wedge \psi(\bar{m}, \bar{c})$ , so  $\mathfrak{U} \models \neg\phi(\bar{m}, \bar{b}')$ , and so  $\text{tp}(\bar{b}/M) \neq \text{tp}(\bar{b}'/M)$ .

(3) Let  $\phi(\bar{x}, \bar{y}; \bar{c}) \in \text{tp}_{\exists}(\bar{a}\bar{b}/C)$ . Since  $\text{tp}_{\exists}(\bar{a}/\bar{b}C)$  is finitely satisfied in  $M$ , there is  $\bar{m} \subset M$  such that  $\mathfrak{U} \models \phi(\bar{m}, \bar{b}; \bar{c})$ . Then since  $\text{tp}_{\exists}(\bar{b}/C)$  is finitely satisfied in  $M$ , there is  $\bar{m}' \subset M$  such that  $\mathfrak{U} \models \phi(\bar{m}, \bar{m}'; \bar{c})$ .  $\square$

One of the key facts about finitely satisfiable types in the usual setting for complete theories is that they are precisely the average types of ultrafilters. From this it follows that if  $p$  is a type over  $A$  that is finitely satisfied in  $M$ , then for any  $B \supset A$ , there is an extension of  $p$  to a type over  $B$  that is still finitely satisfied in  $M$ . However, the average existential type of an ultrafilter need not be maximal existential, and so both of these facts fail in the existentially closed setting of this subsection. Because of this, we will have to leave the setting of existentially closed models for some arguments.

**2.2. Generalized indiscernibles indexed by ordered pairing functions.** Our later results giving consequences of a pre-coding configuration are aided by generalized indiscernibles indexed by the following structure.

**Definition 2.14.** Let  $L_0 = \{I, J, \Gamma, \pi_1, \pi_2, \rho, \leq\}$  and let  $\mathcal{P}$  denote the countable  $L_0$ -structure satisfying:

- (1)  $I, J, \Gamma$  partition  $\mathcal{P}$  into three (disjoint) sorts, each infinite;
- (2)  $\leq$  is dense linear order (DLO) with  $I \ll J \ll \Gamma$  (so the restriction to each sort is also DLO);
- (3)  $\pi_1 : \Gamma \rightarrow I$ ,  $\pi_2 : \Gamma \rightarrow J$ , and  $\rho : I \times J \rightarrow \Gamma$ ;
- (4) For  $\gamma \in \Gamma$ ,  $\rho(\pi_1(\gamma), \pi_2(\gamma)) = \gamma$ ; and
- (5) For  $i \in I, j \in J$ ,  $\pi_1(\rho(i, j)) = i$  and  $\pi_2(\rho(i, j)) = j$ .
- (6)  $\leq \upharpoonright_\Gamma$  is  $\leq_{lex}$ , i.e.,  $\gamma \leq \gamma'$  iff either  $\pi_1(\gamma) < \pi_1(\gamma')$  or  $[\pi_1(\gamma) = \pi_1(\gamma')$  and  $\pi_2(\gamma) \leq \pi_2(\gamma')]$ .

Note that  $(I, \leq)$  and  $(J, \leq)$  are each isomorphic to  $(\mathbb{Q}, \leq)$  and are order-indiscernible sequences in  $\mathcal{P}$ . Moreover, the automorphism group  $Aut(\mathcal{P})$  is naturally isomorphic to  $Aut(I, \leq) \times Aut(J, \leq)$ . Indeed, for any  $\sigma \in Aut(I, \leq)$  and  $\tau \in Aut(J, \leq)$ , we obtain an  $L_0$ -elementary bijection of  $\Gamma$  via  $\rho(i, j) \mapsto \rho(\sigma(i), \tau(j))$ .

Additionally,  $\mathcal{P}$  is uniformly locally finite. As notation, for a finite, non-empty  $X \subseteq \mathcal{P}$ , let  $\langle X \rangle \in Age(\mathcal{P})$  be the smallest substructure containing  $X$ . As  $\mathcal{P}$  is totally ordered, for any finite substructures  $A, B \subseteq \mathcal{P}$ , there is at most one isomorphism  $h : A \rightarrow B$ . If  $\delta_1 < \dots < \delta_k$ ,  $\delta'_1 < \dots < \delta'_k$  are from  $\mathcal{P}$ , then  $\text{qftp}^{\mathcal{P}}(\delta_1, \dots, \delta_k) = \text{qftp}^{\mathcal{P}}(\delta'_1, \dots, \delta'_k)$  if and only if the substructures  $\langle \delta_1, \dots, \delta_k \rangle$  and  $\langle \delta'_1, \dots, \delta'_k \rangle$  are isomorphic.

The following lemma demonstrates that  $\mathcal{P}$  is a desirable index structure.

**Lemma 2.15.** *Age( $\mathcal{P}$ ) has the Ramsey property and  $\mathcal{P}$  is its Fraïssé limit.*

*Proof.* That  $\mathcal{P}$  is homogeneous follows from the natural isomorphism of  $Aut(\mathcal{P})$  with  $Aut(I, \leq) \times Aut(J, \leq)$ . Similarly, by associating a finite substructure of  $\mathcal{P}$  with the product of its projections to  $(I, \leq)$  and  $(J, \leq)$ , the Ramsey property becomes an instance of the product Ramsey theorem [8, §5.1, Theorem 5].  $\square$

As  $\mathcal{P}$  is a Fraïssé limit, any  $L_0$ -isomorphism of finite substructures of  $\mathcal{P}$  extends to an automorphism of  $\mathcal{P}$ .

**Definition 2.16.** Suppose  $L$  is any language and  $M$  is any  $L$ -structure. A subset  $\mathcal{A} \subseteq M$  is  $\mathcal{P}$ -indiscernible via  $f$  if  $f : \mathcal{P} \rightarrow M^{<\omega}$  and  $\mathcal{A} = \bigcup f(\mathcal{P})$  satisfies:

$$\text{tp}_L(f(\delta_1), \dots, f(\delta_k)) = \text{tp}_L(f(\delta'_1), \dots, f(\delta'_k))$$

for all  $k$  and all  $(\delta_1, \dots, \delta_k), (\delta'_1, \dots, \delta'_k)$  from  $\mathcal{P}^k$  satisfying the same quantifier free type in  $\mathcal{P}$ .

We routinely write  $\{\bar{a}_i : i \in I\}$ ,  $\{\bar{b}_j : j \in J\}$ , and  $\{\bar{c}_{i,j} : (i, j) \in I \times J\}$  as the images under  $f$ . We say  $\mathcal{A}$  is a  $\mathcal{P}$ -indiscernible partition if  $f(\delta)$  is without repetition and  $f(\delta) \cap f(\delta') = \emptyset$  for distinct  $\delta, \delta' \in \mathcal{P}$ .

Combining our remarks above, an equivalent of the indiscernibility condition is that for any automorphism  $\sigma \in Aut(\mathcal{P})$ ,  $\text{tp}_L(f(\delta_1), \dots, f(\delta_k)) = \text{tp}_L(f(\sigma(\delta_1)), \dots, f(\sigma(\delta_k)))$  for all  $(\delta_1, \dots, \delta_k) \in \mathcal{P}^k$ . Thus, if in addition,  $M$  is uncountable and saturated, this condition is also equivalent to: For every

$\sigma \in \text{Aut}(\mathcal{P})$ , the set mapping  $f(\delta) \mapsto f(\sigma(\delta))$  extends to an automorphism  $\sigma^* \in \text{Aut}(M)$ .

We now record some technical remarks that will be useful in later sections.

**Remark 2.17.** If  $\mathcal{A}$  is a  $\mathcal{P}$ -indiscernible partition then there are constants  $m_a, m_b, m_\Gamma$  such that  $|f(i)| = m_a$ ,  $|f(j)| = m_b$ , and  $|f(\gamma)| = m_\Gamma$  for all  $i \in I$ ,  $j \in J$ , and  $\gamma \in \Gamma$ . Thus, if  $\mathcal{A}$  is a  $\mathcal{P}$ -indiscernible partition, then  $\mathcal{A}$  is partitioned into  $m_a + m_b + m_c$  ‘strips’, indexed by  $I$ ,  $J$ , or  $I \times J$ . Let  $\mathfrak{C}^+$  denote the monadic expansion of  $\mathfrak{C}$  formed by adding unary predicates  $U_\ell$  for each of these strips. Note that because of our equivalent formulation of  $\mathcal{P}$ -indiscernibility in terms of automorphisms, it follows that  $\mathcal{A}$  is also  $\mathcal{P}$ -indiscernible in the expanded language  $L^+$  of  $\mathfrak{C}^+$ .

**Remark 2.18.** If  $I_0 \subseteq I$  and  $J_0 \subseteq J$  are each dense without endpoints (but not necessarily dense in  $I$  or  $J$ ) then the substructure  $\mathcal{P}_0 \subseteq \mathcal{P}$  with universe  $I_0 \cup J_0 \cup \{\gamma_{i,j} : i \in I_0, j \in J_0\}$  is isomorphic to  $\mathcal{P}$  and every automorphism of  $\mathcal{P}_0$  extends to an automorphism of  $\mathcal{P}$ . Thus,  $\mathcal{A}_0 := f[\mathcal{P}_0]$  is also  $\mathcal{P}$ -indiscernible via  $f$ . If,  $I_0 \subseteq I$  and  $J_0 \subseteq J$  are convex as well, then every automorphism of  $\mathcal{P}_0$  extends to an automorphism of  $\mathcal{P}$  fixing  $(I \setminus I_0) \cup (J \setminus J_0)$  pointwise. In this case  $\mathcal{A}_0$  will be  $\mathcal{P}$ -indiscernible in any expansion of  $\mathfrak{C}$  formed by naming constants from  $\{\bar{a}_i : i \in I \setminus I_0\} \cup \{\bar{b}_j : j \in J \setminus J_0\}$ .

**Remark 2.19.** Suppose  $\mathcal{A}$  is a  $\mathcal{P}$ -indiscernible partition. As the linear order  $(2 \times \mathbb{Q}, \leq)$  embeds into  $(I, \leq)$ , choose disjoint, dense  $I_0, I_1 \subseteq I$  such that for each  $i \in I_1$  there is a unique  $i^- \in I_0$  that is an immediate predecessor of  $i$  and dually, for each  $i^- \in I_0$  there is a unique immediate successor  $i \in I_1$ . For each  $i \in I_1$ , put  $\bar{a}_i^* := \bar{a}_{i^-} \bar{a}_i$ , where  $i^-$  is the immediate predecessor of  $i$ . Then  $\mathcal{A}' = \{\{\bar{a}_i^* : i \in I_1\}, \{\bar{b}_j : j \in J\}, \{\bar{c}_{i,j} : i \in I_1, j \in J\}\}$  is a  $\mathcal{P}_1$ -indiscernible partition, where  $\mathcal{P}_1$  is indexed by  $I_1, J, I_1 \times J$ . By symmetry, one can do the same procedure on the  $J$ -side as well.

### 3. CONFIGURATIONS DEFINED BY LOW-COMPLEXITY FORMULAS

**Notation 3.1.** Throughout this section, we will work with a universal theory  $T$ . We will use  $\mathfrak{U}$  to denote an existentially closed monster model and  $\mathfrak{C}$  to denote a standard saturated monster model.

**3.1. Monadic NIP.** We now aim to prove Theorem 1.2. The first part of the following definition from [15] is the central characterization of monadically NIP theories in [5].

**Definition 3.2.** A theory  $T$  has the *f.s. dichotomy* if for every  $N \models T$ ,  $M \prec N$ ,  $\bar{a}, \bar{b} \subset N$  such that  $\text{tp}(\bar{b}/M\bar{a})$  is finitely satisfiable in  $M$ , and  $c \in N$ , either  $\text{tp}(\bar{b}c/M\bar{a})$  or  $\text{tp}(\bar{b}/M\bar{a}c)$  is finitely satisfiable in  $M$ .

A universal theory  $T$  has the *e-f.s. dichotomy* if for every existentially closed  $N \models T$ ,  $M \prec_{\exists} N$ ,  $\bar{a}, \bar{b} \subset N$  such that  $\text{tp}_{\exists}(\bar{b}/M\bar{a})$  is finitely satisfiable in  $M$ , and  $c \in N$ , either  $\text{tp}_{\exists}(\bar{b}c/M\bar{a})$  or  $\text{tp}_{\exists}(\bar{b}/M\bar{a}c)$  is finitely satisfiable in  $M$ .

Our first subgoal is to show that if  $T$  has the  $e$ -f.s. dichotomy, then  $T$  is monadically NIP. We begin by introducing a characterization of monadic NIP considering arbitrary models of a theory. Then we prove a parallel result in the existentially closed setting assuming the  $e$ -f.s. dichotomy, and finally show how to link them.

**Definition 3.3.** Given  $M \prec_{\exists} N$  existentially closed models of  $T$  (resp.  $M \prec N$  models of  $T$ ), an  $e$ - $M$ -f.s. sequence (resp.  $M$ -f.s. sequence) is a sequence of sets  $(A_i \subset N : i \in I)$  such that  $\text{tp}_{\exists}(A_i/MA_{<i})$  (resp.  $\text{tp}(A_i/MA_{<i})$ ) is finitely satisfiable in  $M$ , for every  $i \in I$ .

Suppose  $X \subseteq N$  is any set.

- A *partial  $e$ - $M$ -f.s. decomposition* of  $X$  is an  $M$ -f.s. sequence  $(A_i : i \in I)$  with  $\bigcup_{i \in I} A_i \subseteq X$ .
- An  *$e$ - $M$ -f.s. decomposition* of  $X$  is a partial  $M$ -f.s. decomposition with  $\bigcup_{i \in I} A_i = X$ .

**Definition 3.4.** For any  $N \models T$  and  $A \subseteq N$ , let  $\text{rtp}(N, A)$  (resp.  $\text{rtp}_{qf}(N, A)$ ,  $\text{rtp}_{\exists}(N, A)$ ) denote the number of complete types (resp. quantifier-free types, existential types) over  $A$  realized by tuples in  $(N \setminus A)^{<\omega}$ .

**Definition 3.5.** Let  $N$  be a structure and let  $\mathcal{I} = \{\bar{a}_i : i \in I\}$  be any sequence of pairwise disjoint tuples in  $N$ . An  $\mathcal{I}$ -partition of  $N$  is any partition  $N = \bigsqcup \{A_i : i \in I\}$  such that  $\bar{a}_i \subseteq A_i$  for each  $i \in I$ .

We will ultimately use the following characterization of monadic NIP. Its crucial feature is that it is in terms of quantifier-free types.

**Proposition 3.6.** *A theory  $T$  is monadically NIP if and only if there is a cardinal  $\lambda(T)$  such that for every  $N \models T$  and every indiscernible sequence  $\mathcal{I} = (\bar{a}_i : i \in I)$  in  $N$  such that  $I$  is a well-ordering with a maximum element, there is an  $\mathcal{I}$ -partition  $(A_i : i \in I)$  of  $N$  such that  $\text{rtp}_{qf}(N, A_{<i}) \leq \lambda(T)$  for every  $i \in I$  (equivalently,  $\text{rtp}(N, A_{<i}) \leq \lambda(T)$  for every  $i \in I$ ).*

Furthermore, for both  $\text{rtp}_{qf}$  and  $\text{rtp}$ , we may take  $\lambda(T) = 2^{|T|}$ .

*Proof.* ( $\Rightarrow$ ) If  $T$  is monadically NIP, then it has the f.s. dichotomy by [5, Theorem 1.1]. As in the proof of [5, Lemma 4.4], we may find a model  $M$  of size  $|T|$  and an  $\mathcal{I}$ -partition  $(A_i : i \in I)$  of  $N$  that is also an  $M$ -f.s. sequence. Then every type realized in  $A_{\geq i}$  over  $A_{<i}$  is finitely satisfiable in  $M$ . The general bound on the number of global types finitely satisfiable in  $M$  is  $\beth_2(|T|)$ , but [16, Proposition 2.43] improves this to  $2^{|T|}$  under the assumption of NIP.

( $\Leftarrow$ ) Let  $\lambda$  be an arbitrary cardinal. By [5, Lemma 4.7], it suffices to show that if  $T$  has IP, rather than merely not being monadically NIP, then we may find  $\mathcal{I}$  and  $N$  such that for every  $\mathcal{I}$ -partition  $(A_i : i \in I)$  of  $N$ , there is an  $i \in I$  with  $\text{rtp}_{qf}(N, A_{<i}) \geq \lambda$ . By [5, Lemma 4.6], it instead suffices to show  $\text{rtp}(N, A_{<i}) \geq \lambda$ .

Assume  $T$  has IP, so by [17] there is a formula (with parameters)  $\phi(x, y)$  on singletons witnessing IP. Let  $N \models T$  contain an indiscernible sequence



$\mathcal{I} = (a_i : i \leq \lambda)$  shattered by  $\phi$ , i.e. there is a set  $Y = \{b_s : s \in 2^\lambda\} \subset N$  such that  $N \models \phi(b_s, a_i) \iff i \in s$ . Consider an  $\mathcal{I}$ -partition  $(A_i : i \leq \lambda)$  of  $N$ . By pigeonhole and the fact that the cofinality of  $2^\lambda$  is greater than  $\lambda$ , there is some  $i^* \in I$  such that  $A_{i^*}$  contains  $Y' \subset Y$  of size  $2^\lambda$ .

Partition  $Y'$  according to the  $\phi$ -type of each element over  $\mathcal{I}_{\leq a_{i^*}}$ . If this partition has at least  $\lambda$  classes then so does the partition using  $\phi$ -types over  $\mathcal{I}_{< a_{i^*}}$ , giving  $\text{rtp}(A_{i^*}, A_{< i^*}) \geq \lambda$ . So suppose it has fewer classes. Then by pigeonhole there is some  $Y^* \subset Y'$  such that  $|Y^*| = 2^\lambda$  and  $\text{tp}_\phi(b/\mathcal{I}_{\leq a_{i^*}})$  is constant among  $b \in Y^*$ . Thus  $\text{tp}_\phi(b/(\mathcal{I}_{> a_{i^*}})) \neq \text{tp}_\phi(b'/(\mathcal{I}_{> a_{i^*}}))$  for distinct  $b, b' \in Y^*$ . So there is  $\mathcal{I}^* \subset \mathcal{I}_{> a_{i^*}}$  such that  $|\mathcal{I}^*| = \lambda$  and  $\text{tp}_{\phi^{\text{opp}}}(a/Y^*) \neq \text{tp}_{\phi^{\text{opp}}}(a'/Y^*)$  for distinct  $a, a' \in \mathcal{I}^*$ . Thus  $\text{rtp}(A_{> i^*}, A_{i^*}) \geq \lambda$ .  $\square$

We will now work toward proving the forward direction of the characterization above in the existentially closed setting, assuming the  $e$ -f.s. dichotomy.

**Lemma 3.7.** *If  $T$  has the  $e$ -f.s. dichotomy, then for every existentially closed  $N \models T$  and  $M \prec_{\exists} N$ , every partial  $e$ - $M$ -f.s. decomposition of  $X \subset N$  can be extended to an  $e$ - $M$ -f.s. decomposition of  $X$ .*

*Proof.* Exactly as in [5, Lemma 3.2], replacing complete types in  $\mathfrak{C}$  with existential types in  $N$ .  $\square$

**Definition 3.8.** A sequence of tuples in a structure is  $e$ -indiscernible if it is indiscernible with respect to all existential formulas.

We will need the following standard result for producing indiscernibles, based on the Erdős-Rado theorem. See [7, Proposition 1.6] for a proof.

**Fact 3.9.** *Let  $T$  be a complete theory,  $\mathfrak{C}$  a monster model for  $T$ , and  $A \subset T$ . If  $\kappa \geq |T| + |A|$  and  $\lambda = \beth_{(2^\kappa)^+}$ , then for any sequence  $(\bar{a}_i : i < \lambda)$ , there is an  $A$ -indiscernible sequence  $(\bar{b}_i : i < \omega)$  such that for each  $n < \omega$ , there are  $i_0, \dots, i_n < \lambda$  such that  $\text{tp}(\bar{b}_0, \dots, \bar{b}_n/A) = \text{tp}(\bar{a}_{i_0}, \dots, \bar{a}_{i_n}/A)$ .*

**Lemma 3.10.** *Let  $T$  be a universal theory. Let  $M \models T$  and let  $\mathcal{I} = (\bar{a}_i : i \in I)$  be an indiscernible sequence (for complete types) in  $M$ . Then there is some  $N \supset M$  such that  $N$  is an existentially closed model of  $T$  and such that  $\mathcal{I}$  is  $e$ -indiscernible in  $N$ .*

*Proof.* We first note it suffices to find some  $|M|^+$ -saturated  $M^* \succ M$  containing an indiscernible sequence  $\mathcal{J}$  of the same order type as  $\mathcal{I}$  and with  $\text{tp}(\mathcal{J}) = \text{tp}(\mathcal{I})$ , and to find an existentially closed  $N^* \supset M^*$  such that  $N^* \models T$  and  $\mathcal{J}$  is  $e$ -indiscernible in  $N^*$ . For then we may find an elementary embedding  $f: M \rightarrow M^*$  with  $f(\mathcal{I}) = \mathcal{J}$ , which restricts to an isomorphism between  $M$  and  $f(M)$ . Since  $N^*$  is a suitable extension of  $f(M)$ , we may find a suitable  $N \supset M$ .

So we now aim to find  $M^*, \mathcal{J}$ , and  $N^*$  as above. Let  $\mu := 2^{|Th(M)|}$ . Choose  $M_1 \succ M$  containing  $\mathcal{I}_1$  of order type  $\beth_{\mu^+}$  and with  $\text{tp}(\mathcal{I}_1) = \text{tp}(\mathcal{I})$ . Let  $N_1 \supset M_1$  be an existentially closed model of  $T$ . We now expand  $N_1$  by a unary predicate naming  $M_1$ , and consider the theory of the pair  $(N_1, M_1)$ .

By Fact 3.9, there is  $(N_2, M_2) \succ (N_1, M_1)$  containing an indiscernible sequence  $\mathcal{J}_0 = (\bar{b}_i : i \in \omega)$  such that for every  $n \geq 1$ ,  $\text{tp}_{(N_2, M_2)}(\bar{b}_1, \dots, \bar{b}_n) = \text{tp}_{(N_1, M_1)}(\bar{a}_{i_1}, \dots, \bar{a}_{i_n})$  for some  $i_1, \dots, i_n \in \beth_{\mu^+}$ . Then let  $(N_3, M^*) \succ (N_2, M_2)$  be  $|M|^+$ -saturated and extending  $\mathcal{J}_0$  to an indiscernible sequence  $\mathcal{J}$  of the same order type as  $\mathcal{I}$ . Finally, let  $N^* \supset N_3$  be an existentially closed model of  $T$ .

By construction,  $\mathcal{J}$  is indiscernible in  $(N_3, M^*)$  and thus in both  $N_3$  and  $M^*$  separately. Furthermore, the  $e$ -type of any tuple from  $\mathcal{J}$  is maximal when considered in  $N_3$ , since this was true of any tuple from  $\mathcal{I}$  considered in  $N_1$  and thus any tuple from  $\mathcal{J}_0$  considered in  $N_2$  and  $\mathcal{J}$  in  $N_3$ . Thus it remains true of any tuple from  $\mathcal{J}$  considered in  $N^*$ . So since  $\mathcal{J}$  was  $e$ -indiscernible in  $N_3$ , it remains  $e$ -indiscernible in  $N^*$ .  $\square$

The following proof is an adaptation of [15, Lemma 4.1] to the existentially closed setting.

**Lemma 3.11.** *Suppose  $\mathcal{I} = (\bar{a}_i : i \in I) \subset \mathfrak{U}$  is  $e$ -indiscernible over  $\emptyset$ . Then there is an existentially closed  $\mathfrak{W} \succ_{\exists} \mathfrak{U}$  and  $M \prec_{\exists} \mathfrak{W}$  with  $|M| = |I|$  such that  $(\bar{a}_i : i \in I)$  is both  $e$ -indiscernible over  $M$  and is an  $e$ - $M$ -f.s. sequence.*

*Proof.* Let  $\mathcal{L}$  be the original language. We first expand the language to  $\mathcal{L}^+$  so the expanded theory has Skolem functions for all existential formulas, and consider the corresponding expansion  $\mathfrak{U}^+$ . Then there is  $\mathfrak{C}^+ \succ \mathfrak{U}^+$  and an indiscernible sequence  $\mathcal{J}^+ = (\bar{b}_i : i \in I)$  in  $\mathfrak{C}^+$  such that for the  $\mathcal{L}$ -reduct  $\mathcal{J}$ ,  $\text{tp}_{\exists}(\mathcal{J}) = \text{tp}_{\exists}(\mathcal{I})$ . By Lemma 3.10, there is  $\mathfrak{W}^+ \supset \mathfrak{C}^+$  an existentially closed monster model of  $\text{Th}(\mathfrak{C}^+)_{\forall}$  such that  $\mathcal{J}^+$  is  $e$ -indiscernible in  $\mathfrak{W}^+$ . Letting  $\mathfrak{W}$  be the  $\mathcal{L}$ -reduct of  $\mathfrak{W}^+$ , there is  $\sigma \in \text{Aut}(\mathfrak{W})$  such that  $\sigma(\mathcal{J}) = \mathcal{I}$ .

Now, in  $\mathfrak{W}^+$ , end-extend  $\mathcal{J}^+$  to obtain  $\mathcal{J}^*$  an  $e$ -indiscernible sequence with order-type  $I + \omega^*$ . Let  $N^+$  be the Skolem hull (with respect to our Skolem functions for existential formulas) of the end-extension, let  $N$  be its  $\mathcal{L}$ -reduct, and let  $M = \sigma(N)$ . We claim  $M$  is as desired.

First note that  $N \prec_{\exists} \mathfrak{W}$ . Then it suffices to show that  $\mathcal{J}$  is  $e$ -indiscernible over  $N$  and is an  $e$ - $N$ -f.s. sequence. Let  $\phi(\bar{x}; \bar{y}; \bar{n})$  be an existential  $\mathcal{L}$ -formula with  $\bar{n} \subset N$  and such that there are  $i_1 < \dots < i_m < j_1 < \dots < j_n \in I$  such that, letting  $\bar{b}_I = \bar{b}_{i_1} \dots \bar{b}_{i_m}$  and  $\bar{b}_J = \bar{b}_{j_1} \dots \bar{b}_{j_n}$ , we have  $\mathfrak{W} \models \phi(\bar{b}_J; \bar{b}_I; \bar{n})$ . Choose  $\bar{b}_{k_1} < \dots < \bar{b}_{k_\ell}$  from the end-extension in  $\mathcal{J}^*$  so that the elements of  $\bar{n}$  can be written as  $\mathcal{L}^+$ -terms using these elements. Finally, choose  $j'_1 < \dots < j'_n < k_1$  also from the end-extension of  $\mathcal{J}^*$ , and let  $\bar{b}_{J'} = \bar{b}_{j'_1} \dots \bar{b}_{j'_n}$ . Then  $\mathfrak{W} \models \phi(\bar{b}_{J'}; \bar{b}_I; \bar{n})$ . Considering elements of  $N$  as  $\mathcal{L}^+$ -terms from the end-extension of  $\mathcal{J}^*$  also quickly gives  $e$ -indiscernibility of  $\mathcal{J}$  over  $N$ .  $\square$

**Lemma 3.12.** *If  $T$  has the  $e$ -f.s. dichotomy, then for every existentially closed  $N \models T$  and every  $e$ -indiscernible sequence  $\mathcal{I} = (\bar{a}_i : i \in I)$  in  $N$  such that  $I$  is a well-ordering with a maximum element, there is an  $\mathcal{I}$ -partition  $(A_i : i \in I)$  of  $N$  such that  $\text{rtp}_{\exists}(N, A_{<i}) \leq \beth_2(|I|)$  for every  $i \in I$ .*

*Proof.* By Lemma 3.11, there is  $\mathfrak{U} \succ_{\exists} N$  and  $M \prec_{\exists} \mathfrak{U}$  such that  $|M| = |I|$  and  $\mathcal{I}$  is an  $e$ - $M$ -f.s. sequence. By Lemma 3.7, we may extend this to an

$e$ - $M$ -f.s. decomposition  $(A_i : i \in I)$  of  $N$ . By Fact 2.13,  $\text{tp}_\exists(A_{\geq i}, A_{< i})$  does not  $e$ -split over  $M$ . As there are at most  $\beth_2(|T|)$  many global existential types that do not  $e$ -split over  $M$ , the result follows.  $\square$

**Lemma 3.13.** *Let  $T$  be a universal theory with the  $e$ -f.s. dichotomy. Then  $T$  is monadically NIP.*

*Proof.* Let  $M \models T$  and  $\mathcal{I}$  be an indiscernible sequence (for complete types) in  $M$ . By Lemma 3.10, there is  $N \supset M$  an existentially closed model of  $T$  such that  $\mathcal{I}$  remains  $e$ -indiscernible in  $N$ . By Lemma 3.12,  $N$  admits an  $\mathcal{I}$ -partition  $\{B_i : i \in I\}$  such that for every  $i \in I$ ,  $\text{rtp}_{qf}(N, B_{< i}) \leq \text{rtp}_\exists(N, B_{< i}) \leq \beth_2(|T|)$ . Letting  $A_i := B_i \cap M$ , we have that  $\{A_i : i \in I\}$  is an  $\mathcal{I}$ -partition of  $M$  such that for every  $i \in I$ ,  $\text{rtp}_{qf}(M, A_{< i}) \leq \beth_2(|T|)$ . Thus  $T$  is monadically NIP, by Proposition 3.6.  $\square$

Our second subgoal is to show that if  $T$  does not have the  $e$ -f.s. dichotomy, then this is witnessed by a particular configuration (a pre-coding configuration) defined by an existential formula.

The construction of this configuration in [5, Proposition 3.11] (although originally due to Shelah in [15]) is fundamentally based on the ability to extend a finitely satisfiable type over a larger set of parameters while maintaining finite satisfiability. As remarked at the end of Section 2.1, this fails when working in existentially closed models, so we will have to move outside this setting to carry out the construction.

**Lemma 3.14.** *Let  $M \prec_\exists \mathfrak{U}$  and  $\bar{a}, \bar{b}, c \in \mathfrak{U}$  witness a failure of the  $e$ -f.s. dichotomy, witnessed by formulas  $\rho_1(\bar{x}, \bar{y}, z), \rho_2(\bar{x}, \bar{y}, z) \in \text{tp}_\exists(\bar{a}\bar{b}c/M)$  such that neither  $\rho_1(\bar{a}, \bar{y}, c)$  nor  $\rho_2(\bar{a}, \bar{y}, z)$  is satisfiable in  $M$ . Then there is  $\mathfrak{C} \succ M$  such that  $\mathfrak{U} \subset \mathfrak{C}$ , and there are  $\bar{b}', c' \in \mathfrak{C}$  such that  $M, \bar{a}, \bar{b}', c'$  witness the failure of the f.s. dichotomy, witnessed by the same formulas  $\rho_1(\bar{x}, \bar{y}, z), \rho_2(\bar{x}, \bar{y}, z) \in \text{tp}^\mathfrak{C}(\bar{a}\bar{b}'c'/M)$  such that neither  $\rho_1(\bar{a}, \bar{y}, c')$  nor  $\rho_2(\bar{a}, \bar{y}, z)$  is satisfiable in  $M$ .*

*Proof.* As  $M$  and  $\mathfrak{U}$  have the same universal theory, there is  $\mathfrak{C} \succ M$  such that  $\mathfrak{U} \subset \mathfrak{C}$ . Note that the existential types of elements in  $\mathfrak{U}$  do not change when  $\mathfrak{U}$  is embedded in  $\mathfrak{C}$ , since they are already maximal.

So, working in  $\mathfrak{C}$ , we have  $p(\bar{y}) := \text{tp}_\exists(\bar{b}/M\bar{a})$  is unchanged (as are all the existential types of elements in  $M$  over  $M\bar{a}$ ) and so it is still finitely satisfiable in  $M$ . Thus  $p(\bar{y})$  can be extended to a complete type  $q(\bar{y})$  that is finitely satisfiable in  $M$  (e.g. by [5, Fact 2.3(2)]). Let  $\bar{b}' \in \mathfrak{C}$  be a realization of  $q$ . Index the elements of  $M^{|\bar{b}'|}$  as  $(\bar{m}_i : i \in I)$ . For each  $i \in I$ , as  $\text{tp}_\exists(\bar{a}\bar{m}_i c)$  is maximal, there must be some existential formula  $\psi_i(\bar{x}, \bar{y}, z)$  in the type that is an obstacle to  $\rho_1$ . For any finite  $I_0 \subset I$ , the formula  $\exists z(\rho_1(\bar{a}, \bar{y}, z) \wedge \rho_2(\bar{a}, \bar{y}, z) \wedge \bigwedge_{i \in I_0} \psi_i(\bar{a}, \bar{m}_i, z))$  is in  $p \subset q$ . So  $r(z) = \{\rho_1(\bar{a}, \bar{b}', z) \wedge \rho_2(\bar{a}, \bar{b}', z) \wedge \psi_i(\bar{a}, \bar{m}_i, z) \mid i \in I\}$  is finitely satisfiable in  $\mathfrak{C}$ . Thus  $r(z)$  is realized in  $\mathfrak{C}$  and we may choose any realization as  $c'$ .  $\square$

**Definition 3.15.** A *pre-coding configuration* in a structure  $M$  consists of a formula  $\phi(\bar{x}, \bar{y}, z)$  with parameters, a sequence  $\mathcal{I} = (d_{\bar{a}i} : i \in \mathbb{Q})$  indiscernible over the parameters of  $\phi$ , and for all  $s < t \in \mathbb{Q}$ , a singleton  $c_{s,t}$  such that the following holds.

- (1)  $M \models \phi(\bar{d}_s, \bar{d}_t, c_{s,t})$
- (2)  $M \models \neg\phi(\bar{d}_s, \bar{d}_v, c_{s,t})$  for all  $v > t$
- (3)  $M \models \neg\phi(\bar{d}_u, \bar{d}_t, c_{s,t})$  for all  $u < s$

A theory *admits pre-coding* if some model contains a pre-coding configuration.

**Lemma 3.16.** *If a universal theory  $T$  does not have the e-f.s. dichotomy, then it admits pre-coding by an existential formula.*

*Proof.* Suppose  $T$  does not have the e-f.s. dichotomy, and let  $\mathfrak{U} \models T$  be an existentially closed model such that  $M \prec_{\exists} \mathfrak{U}$  and  $\bar{a}\bar{b}c \subset \mathfrak{U}$  witness its failure, with existential formulas  $\rho_1, \rho_2$  witnessing the required failures of finite satisfiability. Let  $\mathfrak{C} \succ M$  be as in Lemma 3.14. Then  $\mathfrak{C}$  witnesses a failure of the f.s. dichotomy, with  $\rho_1, \rho_2$  still witnessing the required failures of finite satisfiability, and so  $\mathfrak{C}$  admits pre-coding by [5, Proposition 3.11]. By the proof of the cited result, we may take the pre-coding formula to be  $\rho_1 \wedge \rho_2$ , which is existential.  $\square$

**Theorem 3.17.** *For a universal theory  $T$ , the following are equivalent.*

- (1)  $T$  is monadically NIP.
- (2)  $T$  does not admit pre-coding by an existential formula.
- (3)  $T$  has the e-f.s. dichotomy

*In particular, if  $T$  is not monadically NIP then it admits pre-coding by an existential formula.*

*Proof.* (1)  $\Rightarrow$  (2) If  $T$  admits pre-coding (by any formula), then  $T$  is not monadically NIP by [5, Proposition 3.11, Theorem 1.1].

(2)  $\Rightarrow$  (3) This is Lemma 3.16.

(3)  $\Rightarrow$  (1) This is Lemma 3.13.  $\square$

We close this subsection by introducing a configuration that is related to pre-coding, but is easier to manipulate.

**Definition 3.18.** A *split configuration* in a structure  $M$  consists of a quantifier-free formula  $\psi(\bar{x}, \bar{y}, z, \bar{w})$  (possibly with hidden parameters from  $M$ ) together with disjoint, infinite index sets  $(I, \leq)$ ,  $(J, \leq)$ , and infinite sets of tuples  $\{\bar{a}_i : i \in I\}$ ,  $\{\bar{b}_j : j \in J\}$ , and  $\{\bar{c}_{i,j} : (i, j) \in I \times J\}$  from  $M$  satisfying: for all  $i \in I$  and  $j \in J$ , putting  $\phi(\bar{x}, \bar{y}, z) := \exists \bar{w} \psi$  and letting  $c_{i,j}$  denote the 0-th coordinate of  $\bar{c}_{i,j}$ ,

- $M \models \psi(\bar{a}_i, \bar{b}_j, \bar{c}_{i,j})$
- For all  $i^- < i$ ,  $M \models \neg\phi(\bar{a}_{i^-}, \bar{b}_j, c_{i,j})$
- For all  $j^+ > j$ ,  $M \models \neg\phi(\bar{a}_i, \bar{b}_{j^+}, c_{i,j})$

Clearly, any existential pre-coding configuration gives rise to a split configuration by putting  $I := \mathbb{Q}^-$ ,  $J := \mathbb{Q}^+$ ,  $\bar{a}_i := \bar{d}_i$ , and  $\bar{b}_j := \bar{d}_j$ . However, a split configuration is more malleable, as e.g., we do not require  $\text{lg}(\bar{a}_i)$  to equal  $\text{lg}(\bar{b}_j)$ . This freedom is used in the following lemma.

**Lemma 3.19.** *If  $M, \psi, \{\bar{a}_i : i \in I\}, \{\bar{b}_j : j \in J\}, \{\bar{c}_{i,j} : (i,j) \in I \times J\}$  is a split configuration, then there is an elementary extension  $M' \succeq M$  and a  $\mathcal{P}$ -indiscernible split configuration  $\mathcal{A}$  inside  $M'$  (of the same EM-type). Moreover, among all quantifier-free  $\psi$  yielding a split configuration, if we choose one where  $\text{lg}(\bar{w})$  is least possible, and then among those where  $\text{lg}(\bar{x}) + \text{lg}(\bar{y})$  is least possible, then the  $\mathcal{P}$ -indiscernible split configuration is a  $\mathcal{P}$ -indiscernible partition.*

*Proof.* The first sentence is immediate by compactness, given that  $\text{Age}(\mathcal{P})$  has the Ramsey property by Lemma 2.15. For the second sentence, let  $\psi$  satisfy the minimality conditions and suppose  $\mathcal{A}$  is  $\mathcal{P}$ -indiscernible. We argue that  $\mathcal{A}$  is a  $\mathcal{P}$ -indiscernible partition. First, if any  $\bar{a}_i$  had repeated values, then by  $\mathcal{P}$ -indiscernibility every  $\bar{a}_{i'}$  would have the same values repeated. Thus, we could modify  $\psi$  to include this association, contradicting our minimality assumption. Thus, each  $\bar{a}_i$  is without repeats. If  $\bar{a}_i \cap \bar{a}_{i'} \neq \emptyset$ , then by  $\mathcal{P}$ -indiscernibility we would have some coordinate identically constant. Thus, we could add an external parameter for this common value and again get a smaller  $\psi$ . Thus,  $\{\bar{a}_i : i \in I\}$  are pairwise disjoint with no repeated values. Similar reasoning shows the same holds for  $\{\bar{b}_j : j \in J\}$ .

Now we turn to  $\bar{c}_{i,j}$ . First, if there were  $i \neq k \in I$ ,  $j \neq \ell \in J$  such that  $\bar{c}_{i,j} \cap \bar{c}_{k,\ell} \neq \emptyset$ , then by  $\mathcal{P}$ -indiscernibility there would be a constant value which we could remove. Next, if there were  $i \in I$  and  $j \neq \ell \in J$  such that  $\bar{c}_{i,j} \cap \bar{c}_{i,\ell} \neq \emptyset$ , then by  $\mathcal{P}$ -indiscernibility there would be a common value  $c_i^*$  in some coordinate. But then, by  $\mathcal{P}$ -indiscernibility there would be another value  $c_{i'}^*$  for every other  $i' \in I$ . Then we could take the same  $\psi$ , but rearrange the free variables making this coordinate a part of  $\bar{x}$ . This would decrease  $\text{lg}(\bar{w})$ , contradicting our minimality assumption [even though  $\text{lg}(\bar{x})$  is increased]. Similar reasoning shows that  $\bar{c}_{i,j} \cap \bar{c}_{i',j} = \emptyset$  for all  $i, i' \in I$  and  $j \in J$ . Thus,  $\{\bar{c}_{i,j} : (i,j) \in I \times J\}$  are pairwise disjoint.  $\square$

**Corollary 3.20.** *Suppose  $T$  is a universal  $L$ -theory that is not monadically NIP. Then in some expansion  $L' \supseteq L$  by finitely many constants, there is an  $L'$ -structure  $M' \models T$  and a 0-definable, quantifier-free  $\psi(\bar{x}, \bar{y}, z, \bar{w})$  with a partitioned,  $\mathcal{P}$ -indiscernible split configuration  $\mathcal{A} := \bigcup \{ \{\bar{a}_i : i \in I\}, \{\bar{b}_j : j \in J\}, \{\bar{c}_{i,j} : (i,j) \in I \times J\} \}$  via the mapping  $f(i) := \bar{a}_i$ ,  $f(j) := \bar{b}_j$ ,  $f(\gamma_{i,j}) := \bar{c}_{i,j}$*

*Proof.* As  $T$  is not monadically NIP, it admits a pre-coding configuration and hence a split configuration in some model. Among all possible models of  $T$  and all split configurations, fix one that first minimizes  $\text{lg}(\bar{w})$  and among those, minimizes  $\text{lg}(\bar{x}) + \text{lg}(\bar{y})$  in the underlying  $\psi(\bar{x}, \bar{y}, z, \bar{w})$ . Expand  $M$  by adding constants for the hidden parameters of  $\psi$  and apply Lemma 3.19.  $\square$

**3.2. Monadic stability.** The theorem in this section is proved for graph classes in [12, Theorem 1.3]. Although the proof there generalizes immediately to other languages, we give a somewhat different proof.

**Theorem 3.21.** *Let  $T$  be a universal theory that is monadically NIP but not monadically stable. Then there is a single partitioned atomic formula  $\alpha(\bar{x}; \bar{y})$  that witnesses the order property in every theory  $T'$  such that  $T'_\forall \subset T$ .*

*Proof.* Let  $T^*$  be a completion of  $T$  that is not monadically stable. Since  $T^*$  is monadically NIP, it is unstable by [2]. It is standard that stability of formulas is preserved by boolean combinations, so iterating the following claim will produce an atomic formula witnessing the order property in  $T^*$ , and thus in every theory  $T'$  such that  $T'_\forall \subset T^*_\forall$ , since the order property will be given by a sequence of existential sentences stating the atomic formula defines arbitrarily large half-graphs.

*Claim.* Suppose the formula  $\psi(\bar{x}; \bar{y}) := \exists z \phi(\bar{x}; \bar{y}; z)$  has the order property in  $T^*$ . Then either  $\phi(\bar{x}z; \bar{y})$  or  $\phi(\bar{x}; \bar{y}z)$  has the order property in  $T^*$ .

*Proof of Claim.* We work in a large saturated model  $\mathfrak{C}$  of  $T^*$ . Since  $\psi(\bar{x}; \bar{y})$  has the order property, there is an indiscernible sequence  $\mathcal{I} := (\bar{a}_i \bar{b}_i : i \in \mathbb{Z})$  such that  $\mathfrak{C} \models \psi(\bar{a}_i, \bar{b}_j) \iff i < j$ .

First, assume that for every  $n \geq 1$ ,  $\mathfrak{C} \models \exists z \bigwedge_{j=1}^n \phi(\bar{a}_0, \bar{b}_j, z)$ . Under this assumption the indiscernibility of  $\mathcal{I}$  and compactness implies that for every  $i \in \mathbb{Z}$ ,

$$p_i(z) := \{\phi(\bar{a}_i, \bar{b}_j, z) : i < j\} \cup \{\neg\phi(\bar{a}_i, \bar{b}_j, z) : j \leq i\}$$

is consistent. As  $\mathfrak{C}$  is sufficiently saturated, choosing  $c_i$  to realize  $p_i$ , the sequence  $(\bar{a}_i \bar{b}_i c_i)$  witnesses that  $\phi(\bar{x}z; \bar{y})$  has the order property. Similarly, if for every  $n \geq 1$ ,  $\mathfrak{C} \models \exists z \bigwedge_{j=1}^n \phi(\bar{a}_{-j}, \bar{b}_0, z)$ , then we can find a sequence  $(d_j)$  so that  $(\bar{a}_i \bar{b}_i d_i)$  witnesses that  $\phi(\bar{x}; \bar{y}z)$  has the order property.

Finally, assume neither of these properties hold, and we will contradict  $T^*$  being monadically NIP. Choose any  $s < t$  from  $\mathbb{Z}$  and choose  $c \in \mathfrak{C}$  such that  $\mathfrak{C} \models \phi(\bar{a}_s, \bar{b}_t, c)$ . From the negation of the two previous properties, there is a finite set  $F \subseteq \mathbb{Z}$  such that, letting  $J := \mathbb{Z} \setminus F$ , we have  $\mathfrak{C} \models \neg\phi(\bar{a}_s, \bar{b}_j, c)$  for every  $j \neq t$  and  $\mathfrak{C} \models \neg\phi(\bar{a}_i, \bar{b}_t, c)$  for every  $i \neq s$ . That is,  $c$  witnesses the partial type  $\Gamma_{s,t}(z)$  over  $\bigcup\{\bar{a}_j \bar{b}_j : j \in J\}$  asserting

$$\text{for all } (i, j) \in J^2, \quad \mathfrak{C} \models \phi(\bar{a}_i, \bar{b}_j, z) \quad \text{if and only if} \quad (i, j) = (s, t)$$

By the indiscernibility of  $\mathcal{I}$  and the saturation of  $\mathfrak{C}$ , it follows that for every  $i^* < j^*$  there is  $c_{i^*, j^*}$  such that, for all pairs  $(i, j) \in \mathbb{Z}^2$ ,  $\mathfrak{C} \models \phi(\bar{a}_i, \bar{b}_j, c_{i^*, j^*})$  if and only if  $(i, j) = (i^*, j^*)$ . This is a pre-coding configuration, contradicting  $T^*$  being monadically NIP.  $\diamond$

□

## 4. COLLAPSE OF DIVIDING LINES

We now apply the results of the previous section to hereditary classes of finite structures. For hereditary classes, it is common to use more finitary definitions of (monadic) NIP/stability than Definition 2.1. We begin by showing these are equivalent.

**Definition 4.1.** For a formula  $\phi(\bar{x}; \bar{y})$  with its free variables partitioned and a bipartite graph  $G = (I, J, E)$ , we say a structure  $M$  *encodes*  $G$  via  $\phi$  if there are sets  $A = \{\bar{a}_i \mid i \in I\} \subseteq M^{|\bar{x}|}$ ,  $B = \{\bar{b}_j \mid j \in J\} \subseteq M^{|\bar{y}|}$  such that  $M \models \phi(\bar{a}_i, \bar{b}_j) \Leftrightarrow G \models E(i, j)$ .

Given a class  $\mathcal{C}$  of structures,  $\phi$  *encodes*  $G$  in  $\mathcal{C}$  if there is some  $M_G \in \mathcal{C}$  encoding  $G$  via  $\phi$ .

**Notation 4.2.** Given a class  $\mathcal{C}$  of structures, we use  $Th(\mathcal{C}) := \bigcap_{M \in \mathcal{C}} Th(M)$  to denote the common theory of structures in the class.

**Lemma 4.3.** Let  $\mathcal{C}$  be a class of structures. Then  $Th(\mathcal{C})$  is NIP if and only if no formula encodes every finite bipartite graph in  $\mathcal{C}$ , and  $Th(\mathcal{C})$  is stable if and only if no formula encodes every finite half-graph in  $\mathcal{C}$ .

*Proof.* ( $\Rightarrow$ ) Immediate by compactness.

( $\Leftarrow$ ) Suppose that for every partitioned formula  $\phi(\bar{x}; \bar{y})$  there is some bipartite (half) graph  $G_{\phi(\bar{x}; \bar{y})}$  that  $\phi(\bar{x}; \bar{y})$  does not encode in  $\mathcal{C}$ . This may be expressed by a sentence for each  $\phi(\bar{x}; \bar{y})$ , each of which will be in  $Th(\mathcal{C})$ .  $\square$

Lemma 4.3 suggests considering  $Th(\mathcal{C})$  when given a hereditary class  $\mathcal{C}$ . It would also be natural to view  $\mathcal{C}$  as corresponding to a universal theory and to consider  $Th(\mathcal{C})_{\forall}$  instead, which is often an easier theory to work with as we may pass to arbitrary substructures of infinite models. Since  $Th(\mathcal{C})_{\forall} \subseteq Th(\mathcal{C})$ , it may have more models; for example, if  $\mathcal{C}$  is the class of finite linear orders, then  $(\mathbb{Q}, <)$  is a model of  $Th(\mathcal{C})_{\forall}$  but not of  $Th(\mathcal{C})$ , since all models of  $Th(\mathcal{C})$  are discrete. Nevertheless, we shall see in Theorem 4.6 that for deciding whether a hereditary class is (monadically) stable/NIP, it does not matter whether we consider  $Th(\mathcal{C})$  or  $Th(\mathcal{C})_{\forall}$ . This will be used in the next lemma.

**Lemma 4.4.** Let  $\mathcal{C}$  be a hereditary class of relational structures. Then  $Th(\mathcal{C})$  is monadically NIP (resp. monadically stable) if and only if every class  $\mathcal{C}^+$  obtained by expanding the structures in  $\mathcal{C}$  by unary predicates is NIP (resp. stable).

*Proof.* ( $\Leftarrow$ ) Suppose  $Th(\mathcal{C})$  is monadically NIP and let  $\mathcal{C}^+$  be as described. Let  $M^+ \models Th(\mathcal{C}^+)$ , and let  $M$  be its reduct to the original language. Then  $M \models Th(\mathcal{C})$ , and so is monadically NIP, and thus  $M^+$  is NIP.

( $\Rightarrow$ ) Suppose  $Th(\mathcal{C})$  is not monadically NIP. Then there is  $M \models Th(\mathcal{C})$  admitting a unary expansion  $M^+$  that has IP. Let  $\mathcal{C}^+ := Age(M^+)$ , so  $\mathcal{C}^+$  is contained in a unary expansion of  $\mathcal{C}$ . Then  $M^+ \models Th(\mathcal{C}^+)_{\forall}$  has IP, and thus  $Th(\mathcal{C}^+)_{\forall}$  has IP by Theorem 4.6.  $\square$

The main results of this section will follow quickly from the next lemma showing the collapse between NIP and monadic NIP in hereditary classes.

**Lemma 4.5.** *Let  $\mathcal{C}$  be a hereditary class in a relational language. If  $\text{Th}(\mathcal{C})_{\forall}$  is not monadically NIP, then  $\text{Th}(\mathcal{C})$  is not NIP.*

*Proof.* Suppose  $\text{Th}(\mathcal{C})_{\forall}$  is not monadically NIP, and so by Corollary 3.20 there is an expansion  $L' \supseteq L$  by finitely many constants, an  $M' \models \text{Th}(\mathcal{C})_{\forall}$ , a quantifier-free  $L'$ -formula  $\psi(\bar{x}, \bar{y}, z, \bar{w})$  and a partitioned,  $\mathcal{P}$ -indiscernible split configuration indexed by  $\{\bar{a}_i : i \in \mathbb{Q}^-\}$ ,  $\{\bar{b}_j : j \in \mathbb{Q}^+\}$  and  $\{\bar{c}_{i,j} : i \in \mathbb{Q}^-, j \in \mathbb{Q}^+\}$ . Let  $\phi(\bar{x}, \bar{y}, z)$  denote  $\exists \bar{w} \psi(\bar{x}, \bar{y}, z, \bar{w})$ . For each integer  $i \in \mathbb{Z}^-$ , put  $\bar{a}_i^* := \bar{a}_{i-1/4} \bar{a}_i$  and, for each  $j \in \mathbb{Z}^+$ , put  $\bar{b}_j^* := \bar{b}_j \bar{b}_{j+1/4}$ . Also, put  $\bar{x}^* := \bar{x}' \bar{x}$  and  $\bar{y}^* := \bar{y}' \bar{y}$  and put

$$\theta(\bar{x}^*, \bar{y}^*) := \exists z [\phi(\bar{x}, \bar{y}, z) \wedge \neg \phi(\bar{x}', \bar{y}, z) \wedge \neg \phi(\bar{x}, \bar{y}', z)]$$

We will show that  $\text{Th}(\mathcal{C})$  is not NIP by showing that for every finite, bipartite graph  $G = (S, T; E)$  there is a finite substructure  $M'_G \subseteq M'$  encoding  $G$  via the  $L'$ -formula  $\theta(\bar{x}^*, \bar{y}^*)$ .

Let  $m := |\bar{c}_{i,j}| + 1$ . For every integer  $j \in \mathbb{Z}$ , let  $P(j) := \{j + \frac{k}{4m} : -2m < k < 2m\}$ , which is a finite set of rationals in the interval  $(j - 1/2, j + 1/2)$ . For disjoint finite sets of integers  $S \subseteq \mathbb{Z}^-, T \subseteq \mathbb{Z}^+$ , let  $I_{ST} := \bigcup \{P(j) : j \in S \cup T\}$ . Clearly, if  $j < j'$  are integers, then  $P(j) \ll P(j')$ . The salient feature of  $I_{ST}$  is that if  $j \in S \cup T$  and  $Z \subseteq P(j)$  is of size  $\leq m$  with  $j \in Z$ , then there are order-preserving functions  $g, h : Z \rightarrow P(j)$  such that  $g(j) = j + 1/4$  and  $h(j) = j - 1/4$ .

Let  $M'_\emptyset$  be the finite substructure of  $M'$  with universe

$$\bar{p} \cup \{\bar{a}_r : r \in I_{ST}, r < 0\} \cup \{\bar{b}_s : s \in I_{ST}, s > 0\}$$

where  $\bar{p}$  is the finite tuple of  $M'$  named by constants and let  $M'_{ST}$  be the finite substructure of  $M'$  with universe

$$M'_\emptyset \cup \{\bar{c}_{r,s} : r \in \bigcup_{i \in S} P(i), s \in \bigcup_{j \in T} P(j)\}$$

Let  $j \in T$ ,  $Z \subset P(j)$ , and  $f : Z \rightarrow I_{ST}$  be as above. Let  $Z' \subset M_{ST}$  consists of all points with (all) indices in  $Z \cup (I_{ST} \setminus P(j))$ . Then we define  $g' : Z' \rightarrow M_{ST}$  as follows.

- If  $v \in \bar{b}_z$  for  $z \in Z$ , then  $g'(v)$  is the associated element of  $\bar{b}_{g(z)}$ .
- If  $v \in \bar{c}_{r,z}$  for  $z \in Z$ , then  $g'(v)$  is the associated element of  $\bar{c}_{r,g(z)}$ .
- Otherwise,  $g'(v) = v$  (noting that  $g(z) = z$ ).

Coupled with the  $\mathcal{P}$ -indiscernibility, this yields that for any  $m$ -tuple  $\bar{v}$ ,  $\text{tp}^{M'}(\bar{v}) = \text{tp}^{M'}(g'(\bar{v}))$  for any of the functions  $g' : Z' \rightarrow M_{ST}$  described above.

Now, for any bipartite graph  $G = (S, T; E)$ , let  $M'_G \subseteq M_{ST}$  be the substructure with universe

$$M'_\emptyset \cup \bigcup_{(i,j) \in E} \{\bar{c}_{r,s} : r \in P(i), s \in P(j)\}$$



Note that for any  $g'$  as above, its restriction to  $M'_G$  has its image contained within  $M'_G$ . Clearly,  $M'_G$  need not be  $\mathcal{P}$ -indiscernible for all formulas, but it will be  $\mathcal{P}$ -indiscernible for quantifier-free formulas, and so for any  $m$ -tuple  $\bar{v}$  in  $M'_G$ ,  $\text{qftp}^{M'_G}(\bar{v}) = \text{qftp}^{M'_G}(g'(\bar{v}))$ .

*Claim.*  $M'_G \models \theta(\bar{a}_i^*, \bar{b}_j^*)$  if and only if  $(i, j)$  is an edge in  $G$ .

*Proof of Claim.* If  $(i, j)$  is an edge in  $G$ , then  $\bar{c}_{i,j} \subset M'_G$ , so  $\bar{c}_{i,j}^0$  witnesses the outermost existential in  $\theta$ , with the rest of  $\bar{c}_{i,j}$  witnessing the existentials in  $\phi(\bar{a}_i, \bar{b}_j, c_{i,j})$ . Since  $M' \models \neg\phi(\bar{a}_{i-1/4}, \bar{b}_j, c_{i,j}) \wedge \neg\phi(\bar{a}_i, \bar{b}_{j+1/4}, c_{i,j})$  and  $\neg\phi$  is universal, this is true in  $M'_G$  as well.

Now suppose  $(i, j)$  is not an edge in  $G$ . Suppose there are  $e^0 \in M'_G$  and  $\bar{e} \subset M'_G$  such that  $M'_G \models \psi(\bar{a}_i, \bar{b}_j, e^0 \bar{e})$ . To show  $M'_G \models \neg\theta(\bar{a}_i, \bar{b}_j)$  it suffices to find some  $\bar{e}' \subset M'_G$  such that either  $M'_G \models \psi(\bar{a}_{i-1/4}, \bar{b}_j, e^0 \bar{e}')$  or  $M'_G \models \psi(\bar{a}_i, \bar{b}_{j+1/4}, e^0 \bar{e}')$ . Note that  $e^0 \in M'_G$  has at most two indices, and since  $(i, j) \notin E$ , they cannot be in both  $P(i)$  and  $P(j)$ . For definiteness, suppose no index of  $e^0$  is in  $P(j)$ . Let  $Z \subseteq P(j)$  be the indices in  $\bar{b}_j \bar{e}$  contained in  $P(j)$ , and note  $|Z| \leq m$ . Let  $Z'$  and  $g': Z' \rightarrow M_{ST}$  be as above; so  $g'$  is order-preserving on the indices of points, sends points with index  $j$  to  $j + 1/4$ , and fixes points whose indices are outside  $P(j)$ . Thus  $g'(\bar{a}_i \bar{b}_j e^0 \bar{e}) = (\bar{a}_i \bar{b}_{j+1/4} e^0 g'(\bar{e}))$ . By the paragraph before this Claim,  $g'(\bar{e}) \subset M'_G$  and  $M'_G \models \psi(\bar{a}_i, \bar{b}_{j+1/4}, e^0 g'(\bar{e}))$ , which suffices.  $\diamond$

$\square$

**Theorem 4.6.** *Let  $\mathcal{P} \in \{\text{stable}, \text{NIP}\}$ . Let  $\mathcal{C}$  be a hereditary class of relational structures. Then the following are equivalent.*

- (1)  $\text{Th}(\mathcal{C})_{\forall}$  is monadically  $\mathcal{P}$ .
- (2)  $\text{Th}(\mathcal{C})$  is monadically  $\mathcal{P}$ .
- (3)  $\text{Th}(\mathcal{C})_{\forall}$  is  $\mathcal{P}$ .
- (4)  $\text{Th}(\mathcal{C})$  is  $\mathcal{P}$ .

*Proof.* The equivalence of (1) and (2) is by Proposition 2.5, and clearly (1)  $\Rightarrow$  (3)  $\Rightarrow$  (4). So it only remains to show (4)  $\Rightarrow$  (1). We will show the contrapositive in each case.

Case A:  $\mathcal{P} = \text{NIP}$ . This is Lemma 4.5.

Case B:  $\mathcal{P} = \text{stable}$ . Suppose  $\text{Th}(\mathcal{C})_{\forall}$  is not monadically stable, and so neither is  $\text{Th}(\mathcal{C})$ , by Proposition 2.5. By [2], either  $\text{Th}(\mathcal{C})$  is not monadically NIP and thus not even NIP by Case A, or it is monadically NIP but unstable. In either case, we are finished.  $\square$

**Corollary 4.7.** *Let  $T$  be a (possibly incomplete) theory in a relational language, and let  $\mathcal{P} \in \{\text{stable}, \text{NIP}\}$ . Then  $T$  is monadically  $\mathcal{P}$  if and only if  $T_{\forall}$  is  $\mathcal{P}$ .*

*Proof.* From Proposition 2.5, it suffices to show that if  $T_{\forall}$  is  $\mathcal{P}$  then  $T_{\forall}$  is monadically  $\mathcal{P}$ . Let  $\mathcal{C}$  be the hereditary class of finite models of  $T_{\forall}$ , so  $\text{Th}(\mathcal{C})_{\forall} = T_{\forall}$ . The result now follows from Theorem 4.6.  $\square$

The following definition is analogous to monotone graph classes, which are closed under (not-necessarily-induced) subgraph, or equivalently under the removal of vertices and edges.

**Definition 4.8.** A hereditary relational class  $\mathcal{C}$  is a *monotone class* if for every  $M \in \mathcal{C}$ , every structure obtained from  $M$  by removing instances of atomic relations (other than equality and non-equality) is still in  $\mathcal{C}$ .

Note that this definition is not exactly a generalization of monotone graph classes, since for graphs the edge relation must be symmetric, so if we remove the relation  $E(a, b)$  we must also remove  $E(b, a)$ . Placing such additional symmetry constraints on the relations of  $\mathcal{C}$  would not change the proof of the next theorem.

We now generalize part of the main result of [1] from graphs to relational structures, answering part of [11, Problem 5.1].

**Theorem 4.9.** *Let  $\mathcal{C}$  be a monotone class of relational structures. Then  $Th(\mathcal{C})$  is NIP if and only if  $Th(\mathcal{C})$  is monadically stable.*

*Proof.* The backward direction is immediate. For the forward direction, by Theorem 4.6 it suffices to show that if  $Th(\mathcal{C})_{\forall}$  is monadically NIP then it is stable. So suppose it is monadically NIP but unstable, and we will create a contradiction by showing it is not NIP. By Theorem 3.21, there is an atomic formula  $\phi(\bar{x}; \bar{y})$  with the order property in  $Th(\mathcal{C})_{\forall}$ . Let  $(\bar{a}_i : i \in \mathbb{Z})$ ,  $(\bar{b}_j : j \in \mathbb{Z})$  be in  $M \models Th(\mathcal{C})_{\forall}$  such that  $M \models \phi(\bar{a}_i; \bar{b}_j) \iff i < j$ . So  $\phi$  defines a complete bipartite graph on  $\{\bar{a}_i \mid i \in \mathbb{Z}^-\} \times \{\bar{b}_j \mid j \in \mathbb{Z}^+\}$ . As  $\mathcal{C}$  is monotone and  $\phi$  is atomic, we may produce a model of  $Th(\mathcal{C})_{\forall}$  by removing instances of  $\phi$  so the remaining instances define the random bipartite graph. Thus  $Th(\mathcal{C})_{\forall}$  has IP.  $\square$

## 5. MANY MODELS

We close with an application to the finite combinatorics of hereditary classes. We show that if a class  $\mathcal{C}$  of finite relational structures is not monadically NIP then it has superexponential growth rate in the following sense, removing the hypothesis of quantifier elimination from a result in [5].

**Definition 5.1.** Let  $\mathcal{C}$  be a hereditary class. The (*unlabeled*) *growth rate* of  $\mathcal{C}$  is the function  $f_{\mathcal{C}}(n)$  counting the number of isomorphism types in  $\mathcal{C}$  with  $n$  elements.

The class of all graphs with degree at most five is monadically NIP (in fact mutually algebraic, and thus monadically NFCP [9]) and has labeled growth rate  $\Omega(n^{5n/2})$  [13, Formula (6.6)], so its unlabeled growth rate is  $\Omega(n^{3n/2})$ . Thus the converse to our theorem does not hold, and monadic NIP does not separate classes according to their growth rates, since this example's unlabeled growth rate is faster than that of the class of permutations viewed as structures with two linear orders, which is not monadically NIP. However, under the additional assumption that  $\mathcal{C}$  is the class of finite substructures of

an  $\omega$ -categorical structure, we have conjectured that monadic NIP implies  $f_{\mathcal{C}}(n)$  is at most exponential [5, Conjecture 1].

Fast growth rate will be a quick consequence of the following non-structure result, which we isolate for possible further applications. Like Lemma 4.5, we will be encoding bipartite graphs in finite structures, but we allow ourselves to expand the language by unary predicates. Using this, the sets of tuples on which we encode bipartite graphs will be made definable, which then allows us to define the graphs on singletons. This much was already shown in [2, Theorem 8.1.8], but our approach allows us to additionally bound the size of the structure we are using to define a given graph.

**Proposition 5.2.** *Let  $\mathcal{C}$  be a hereditary class in a relational language  $\mathcal{L}$ . If  $\text{Th}(\mathcal{C})$  is not monadically NIP, then there is an expansion  $\mathcal{L}^* \supset \mathcal{L}$  by finitely many unary predicates and a corresponding expansion  $\mathcal{C}^*$  of  $\mathcal{C}$ , and  $\mathcal{L}^*$ -formulas  $U^*(x)$ ,  $V^*(x)$ , and  $E^*(x, y)$  on singletons such that for every finite bipartite graph  $G = (U, V; E)$ , there is  $M_G^* \in \mathcal{C}^*$  such that  $G \cong (U^*(M_G^*), V^*(M_G^*); E^*(M_G^*))$  and  $|M_G^*| = O(|U| + |V| + |E|)$ .*

*Proof.* As  $\text{Th}(\mathcal{C})$  is not monadically NIP, by Corollary 3.20 we can find a saturated model  $\mathfrak{C} \models \text{Th}(\mathcal{C})$  with a partitioned,  $\mathcal{P}$ -indiscernible  $\mathcal{A} \subseteq \mathfrak{C}$  for which there is a quantifier-free  $\psi(\bar{x}, \bar{y}, z, \bar{w})$  such that, letting  $\phi(\bar{x}, \bar{y}, z) := \exists \bar{w} \psi$  and letting  $c_{i,j}^0$  denote the  $0^{\text{th}}$  coordinate of  $\bar{c}_{i,j}$  we have, for all  $i \in I$ ,  $j \in J$ ,

- $\mathfrak{C} \models \psi(\bar{a}_i, \bar{b}_j, \bar{c}_{i,j}^0)$ ;
- $\mathfrak{C} \models \neg \phi(\bar{a}_{i'}, \bar{b}_j, c_{i,j}^0)$  for all  $i' < i$  from  $I$ ; and
- $\mathfrak{C} \models \neg \phi(\bar{a}_i, \bar{b}_{j'}, c_{i,j}^0)$  for all  $j' > j$  from  $J$ .

We now ‘duplicate’  $\mathcal{A}$  according to Remark 2.19 twice, once for  $I$  and once for  $J$ . Letting  $\bar{x}^* := \bar{x}'\bar{x}$  and  $\bar{y}^* := \bar{y}\bar{y}'$  and, for each  $i \in I_1$ , put  $\bar{a}_i^* := \bar{a}_i - \bar{a}_i$  and put  $\bar{b}_j^* := \bar{b}_j \bar{b}_{j+}$  for each  $j \in J_0$ . Also, put

$$\chi(\bar{x}^*, \bar{y}^*, z) := \phi(\bar{x}, \bar{y}, z) \wedge \neg \phi(\bar{x}', \bar{y}, z) \wedge \neg \phi(\bar{x}, \bar{y}', z)$$

We obtain, for each  $i, k \in I_1$  and each  $j, \ell \in J_0$ ,

$$\mathfrak{C} \models \chi(\bar{a}_i^*, \bar{b}_j^*, c_{k,\ell}^0) \quad \text{if and only if} \quad (i, j) = (k, \ell)$$

Next, we look at all specializations  $\chi'(\bar{x}', \bar{y}', z)$  formed by replacing some of the free variables in  $\bar{x}^*$  by constants representing elements of  $\bigcup \{\bar{a}_i^* : i \in I_1\}$  and replacing some of the free variables in  $\bar{y}^*$  by constants representing elements of  $\bigcup \{\bar{b}_j^* : j \in J_0\}$ . Call such a specialization *allowable* if there are convex subsets  $I' \subseteq I_1$ ,  $J' \subseteq J_0$  such that all of the parameters added are from  $\bigcup \{\bar{a}_i^* : i \in I_1 \setminus I'\} \cup \bigcup \{\bar{b}_j^* : j \in J_0 \setminus J'\}$  and  $\mathfrak{C}$  still satisfies

$$\mathfrak{C} \models \chi'(\bar{a}'_i, \bar{b}'_j, c_{k,\ell}^0) \quad \text{if and only if} \quad (i, j) = (k, \ell)$$

for all  $i, k \in I'$ ,  $j, \ell \in J'$ , where  $\bar{a}'_i$  is the restriction of  $\bar{a}_i^*$  to the free variables  $\bar{x}'$  and dually for  $\bar{b}'_j$ .

Clearly,  $\chi$  itself is allowable, taking  $I' = I_1$  and  $J' = J_0$ . Among all such allowable specializations, choose one that minimizes  $\text{lg}(\bar{x}') + \text{lg}(\bar{y}')$  and add these new constant symbols to the language.

After reindexing, replacing  $I_1$  by  $I'$ ,  $J_0$  by  $J'$ ,  $\bar{a}_i^*$  by  $\bar{a}'_i$ , and  $\bar{b}_j^*$  by  $\bar{b}'_j$ , we have (by Remark 2.18 applied to  $I', J'$ ) a partitioned,  $\mathcal{P}$ -indiscernible (in this larger language)  $\mathcal{A}'$  and  $\chi'(\bar{x}', \bar{y}', z)$ , which is a boolean combinations of specializations of  $\phi$  such that

$$\mathfrak{C} \models \chi'(\bar{a}'_i, \bar{b}'_j, \bar{c}_{k,\ell}^0) \quad \text{if and only if} \quad (i, j) = (k, \ell)$$

for all  $i, k \in I'$  and  $j, \ell \in J'$ . The additional minimality property we have gained on  $\chi'$  will be used in the proof of Claim 2. Let  $m_a = \text{lg}(\bar{a}'_i)$ ,  $m_b = \text{lg}(\bar{b}'_j)$ , and  $m_c = \text{lg}(\bar{c}_{i,j})$ .

After doing this minimization, let  $\mathfrak{C}^+$  denote the expansion of  $\mathfrak{C}$  by the unary predicates defining the strips of  $\mathcal{A}'$ . We use  $A^j$  to define the  $j^{\text{th}}$  coordinate strip of  $(\bar{a}_i : i \in I)$  for  $0 \leq j \leq m_a - 1$ , and similarly  $B^j$  and  $C^j$ . By Remark 2.17,  $\mathcal{A}'$  remains  $\mathcal{P}$ -indiscernible in  $\mathfrak{C}^+$  with respect to all  $L^+$ -formulas. Now, to further simplify the notation, remove the primes from all of the items discussed above.

Let  $\chi^+(\bar{x}, \bar{y}, z)$  be the same formula as  $\chi$ , except that in each instance of  $\phi$  or its negation, replace  $\exists \bar{w}$  by  $\exists w_1 \in C^1 \exists w_2 \in C^2 \dots \exists w_{m_c-1} \in C^{m_c-1}$ . Note that we still have

$$\mathfrak{C}^+ \models \chi^+(\bar{a}_i, \bar{b}_j, \bar{c}_{k,\ell}^0) \quad \text{if and only if} \quad (i, j) = (k, \ell)$$

for all  $i, k \in I$ ,  $j, \ell \in J$ .

We now define various subsets of the index sets  $(I, \leq)$  and  $(J, \leq)$ , recalling that both of these are isomorphic to  $(\mathbb{Q}, \leq)$ . Let  $D_I$  be a discrete subset of  $I$  of order type  $\omega$ , and let  $i^*$  denote the least element of  $D_I$ . For each  $i \in D_I$ , choose ‘neighbors’  $i^-, i^+$  such that  $i^- < i < i^+$  and, letting  $Nb(i) = \{i^-, i, i^+\}$ , such that  $Nb(i) \ll Nb(i')$  whenever  $i < i'$  in  $D_I$ .

Put  $3D_I := \bigcup \{Nb(i) : i \in D_I\}$ . For each  $i \in 3D_I$ , let its ‘cilia’  $Cil(i)$  consist of  $2m_c + 1$  points, centered at  $i$ , such that  $Cil(i) \ll Cil(i')$  whenever  $i < i'$  in  $3D_I$ .

Similarly, let  $D_J \subseteq J$  be discrete of order type  $\omega^*$  with largest element  $j^*$ , and define  $N(j)$ ,  $3D_J$ , and  $Cil(j)$  analogously.

Let  $\mathfrak{C}^{++}$  be the expansion of  $\mathfrak{C}^+$  formed by naming each element of  $\bar{a}_{i^*}$  and  $\bar{b}_{j^*}$  by constant symbols and adding the following five unary predicates:

- $E_I := \{c_{i,j^*}^0 : i \in D_I\}$ ;
- $F_I := \{c_{i,j}^0 : i \in 3D_I, j \in Nb(j^*)\}$ ;
- $E_J := \{c_{i^*,j}^0 : j \in D_J\}$ ;
- $F_J := \{c_{i,j}^0 : i \in Nb(i^*), j \in 3D_J\}$ ; and
- $N := \{\bar{a}_i : i \in D_I\} \cup \{\bar{b}_j : j \in D_J\} \cup \{\bar{c}_{i,j} : i \in Cil(3D_I), j \in Nb(j^*)\} \cup \{\bar{c}_{i,j} : i \in Nb(i^*), j \in Cil(3D_J)\}$

We also define two  $L^{++}$ -formulas

$$\alpha(\bar{x}) := \bigwedge_{\ell < m_a} A^\ell(x_\ell) \wedge \exists z (E_I(z) \wedge \chi^+(\bar{x}, \bar{b}_{j^*}, z)) \wedge \forall z [(\chi^+(\bar{x}, \bar{b}_{j^*}, z) \wedge F_I(z)) \rightarrow E_I(z)]$$

$$\beta(\bar{y}) := \bigwedge_{\ell < m_b} B^\ell(y_\ell) \wedge \exists z (E_J(z) \wedge \chi^+(\bar{a}_{i^*}, \bar{y}, z)) \wedge \forall z [(\chi^+(\bar{a}_{i^*}, \bar{y}, z) \wedge F_J(z)) \rightarrow E_J(z)]$$

Now, given any finite sets  $S \subseteq D_I \setminus \{i^*\}$  and  $T \subseteq D_J \setminus \{j^*\}$ , consider the  $L^{++}$ -substructure  $N_{S,T} \subseteq \mathfrak{C}^{++}$  with universe

$$\bar{p} \cup \{\bar{a}_i : i \in S \cup \{i^*\}\} \cup \{\bar{b}_j : j \in T \cup \{j^*\}\} \cup \{\bar{c}_{i,j} : (i,j) \in (Cil(S) \times Nb(j^*)) \cup (Nb(i^*) \times Cil(T))\}$$

where  $\bar{p}$  are the interpretations of the  $L^{++}$ -constant symbols. Note that the cardinality  $|N_{S,T}|$  is  $O(|S| + |T|)$ .

*Claim 1.* For any  $\bar{d} \subset \bigcup \{\bar{a}_i : i \in S\}$  of length  $m_a$  and any  $c \in F_I \cap N_{S,T}$ ,  $\mathfrak{C}^{++} \models \chi^+(\bar{d}, \bar{b}_{j^*}, c)$  if and only if  $N_{S,T} \models \chi^+(\bar{d}, \bar{b}_{j^*}, c)$ .

*Proof of Claim.* As  $\chi^+$  is a boolean combination of existential formulas  $\delta$  in which every existential quantifier is bound to some  $C^\ell$ , it suffices to prove that if  $\mathfrak{C}^{++} \models \exists w_1 \in C^1 \dots \exists w_{m_c-1} \in C^{m_c-1} \rho(\bar{d}, \bar{b}_{j^*}, c, w_1, \dots, w_{m_c-1})$ , where  $\rho$  is a quantifier-free  $L^{++}$ -formula, then the same holds in  $N_{S,T}$ . But this is ensured by the cilia around each point of  $3D_I$  and of  $N(j^*)$  and by the  $\mathcal{P}$ -indiscernibility of  $\mathcal{A}$ . In particular, in  $N_{S,T}$  each  $c \in F_I$  is at the center of a  $(2m_c + 1) \times (2m_c + 1)$  grid of  $\bar{c}$ -tuples arising from the cilia, and by  $\mathcal{P}$ -indiscernibility if there exist witnesses in  $\mathfrak{C}^{++}$  to the existential quantifiers above, then witnesses can be found within these grids.  $\diamond$

*Claim 2.* The  $L^{++}$ -formula  $\alpha(\bar{x})$  defines  $\{\bar{a}_i : i \in S\}$  in  $N_{S,T}$ .

*Proof of Claim.* First, for each  $i \in S$ ,  $\mathfrak{C}^{++} \models \chi^+(\bar{a}_i, \bar{b}_{j^*}, c_{k,\ell})$  if and only if  $(i, j^*) = (k, \ell)$ . Thus, by Claim 1,  $c_{i,j^*}$  is the unique solution to  $\chi^+(\bar{a}_i, \bar{b}_{j^*}, z)$  in  $N_{S,T}$ . As  $E_I(c_{i,j^*})$  holds, we have  $N_{S,T} \models \alpha(\bar{a}_i)$ .

Conversely, assume  $N_{S,T} \models \chi^+(\bar{d}, \bar{b}_{j^*}, z) \wedge E_I(z)$ . By construction, any such  $z$  has the form  $c_{i,j^*}$  for some  $i \in S$ . Now assume  $\bar{d} \neq \bar{a}_i$  (and so  $\bar{d} \neq \bar{a}_k$  for any  $k$ ) and we will show  $N_{S,T} \not\models \chi^+(\bar{d}, \bar{b}_{j^*}, c')$  where  $c'$  is one of the eight points  $c_{i',j'}$  where  $i' \in N(i)$ ,  $j' \in N(j^*)$ , with  $(i', j') \neq (i, j^*)$ . As each of these eight points is in  $F_I$ , we would conclude  $N_{S,T} \models \neg \alpha(\bar{d})$ . To show the missing step, note that by Claim 1, it suffices to prove this in  $\mathfrak{C}^{++}$ . The proof of [5, Lemma 4.11] shows precisely that if  $\chi^+$  satisfies the minimality condition we have imposed on it, then for every  $\epsilon > 0$  there is some  $c_{i',j'}$  such that  $(i', j') \neq (i, j^*)$ ,  $i - \epsilon < i' < i + \epsilon$  and  $j^* - \epsilon < j' < j^* + \epsilon$ , and  $\mathfrak{C}^{++} \models \chi^+(\bar{d}, \bar{b}_{j^*}, c_{i',j'})$ . By  $\mathcal{P}$ -indiscernibility we may take  $c_{i',j'}$  to be one of the eight points described above.  $\diamond$

By symmetric claims and identical reasoning, we conclude that  $\beta(\bar{y})$  defines  $\{\bar{b}_j : j \in T\}$  in  $N_{S,T}$ .

We are now ready to form our finite structures  $M_G^*$  encoding bipartite graphs. Given any finite bipartite graph  $G = (S, T; E)$ , let  $M_G^*$  be the  $\mathfrak{C}^{++}$  substructure with universe

$$\bar{p} \cup N_{S,T} \cup \{\bar{c}_{i,j} : (i, j) \in E\}$$

Visibly,  $|M_G^*|$  is  $O(|S| + |T| + |E|)$ . Since  $N_{S,T}$  is definable in  $M_G^*$  via  $\gamma(x) := N(x) \vee x \in \bar{p}$ , we may define  $\alpha^\gamma(\bar{x})$  as the “ $\gamma$ -relativized” version of  $\alpha$ , by taking  $\alpha$  and requiring that all variables (free and bound) belong to  $\gamma$ , and similarly define  $\beta^\gamma$ . Now, put  $U^*(x) := A^0(x) \wedge x \neq \bar{a}_{i,*}^0$ ,  $V^*(y) := B^0(y) \wedge y \neq \bar{b}_{j,*}^0$ , and  $E^*(x, y) := \exists \bar{x}\bar{y}(\alpha^\gamma(x\bar{x}) \wedge \beta^\gamma(y\bar{y}) \wedge \exists z\chi^+(x\bar{x}, y\bar{y}, z))$ .  $\square$

**Theorem 5.3.** *Let  $\mathcal{C}$  be a hereditary class in a relational language. If  $\text{Th}(\mathcal{C})$  is not monadically NIP, then there is some  $k \in \omega$  such that the unlabeled growth rate  $f_{\mathcal{C}}(n) = \Omega(\lfloor n/k \rfloor!)$ .*

*Proof.* If  $\mathcal{C}$  is not monadically NIP, let  $\mathcal{C}^*$  be an expansion and  $U^*, V^*, E^*$  be formulas as in Proposition 5.2. Given a bipartite graph  $G = (U, V; E)$ , let  $M_G^*$  be as in Proposition 5.2. We have  $|M_G^*| = O(|U| + |V| + |E|)$ , so if  $G$  has  $n$  edges and no isolated vertices, then  $|M_G^*| \leq Kn$  for some fixed  $K \in \omega$ . As shown within Case (a) of the proof of [10, Theorem 1.5], the number of such graphs is  $\Omega(\lfloor n/5 \rfloor!)$ . If  $G \not\cong H$  then  $M_G^* \not\cong M_H^*$ , so  $f_{\mathcal{C}^*}(n) = \Omega(\lfloor n/5K \rfloor!)$ . Having added finitely many unary predicates and named finitely many constants in passing to  $\mathcal{C}^*$  affects the growth rate by at most an exponential factor, so we obtain the desired bound on  $f_{\mathcal{C}}(n)$ .  $\square$

**Remark 5.4.** The optimality of the lower bound in this theorem is witnessed by the family of hereditary classes  $\{\text{Perm}_k \mid k \in \mathbb{Z}^+\}$ , where  $\text{Perm}_k$  encodes permutations on disjoint  $k$ -tuples, i.e. the language of  $\text{Perm}_k$  is two  $2k$ -ary relations, and the structures are obtained by taking a permutation (represented as a structure in the language of two linear orders) and blowing up each point to a  $k$ -tuple with no further structure, and then closing under substructure. By separating those points in a full  $(k+1)$ -tuple from those that are not, the growth rate of  $\text{Perm}_{k+1}$  is seen to be bounded above by  $n \lfloor n/(k+1) \rfloor!$ , which is  $O(\lfloor n/k \rfloor!)$ .

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