An analogue of U-rank for atomic classes

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Abstract

For a countable, complete, first-order theory T, we study $\mathbf{At_T}$, the class of atomic models of T. We develop an analogue of U-rank and prove two results. On one hand, if some $\mathrm{tp}(d/a)$ is not ranked, then there are $2^{\aleph_1}$ non-isomorphic models in $\mathbf{At_T}$ of size $\aleph_1$. On the other hand, if all types have finite rank, then the rank is fully additive and every finite tuple is dominated by an independent set of realizations of pseudo-minimal types.

For a countable, complete first order theory T, a model M is *atomic* if $\mathrm{tp}(a)$ is principal, i.e., is generated by a complete formula for every finite tuple a from M. In this paper, we continue our investigations of dichotomies among classes $\mathbf{At_T}$ of atomic models of a countable, complete, first-order theory T. One reason for studying such classes relates to complete sentences of $L_{\omega_1,\omega}$. It is well known, see e.g., [Bal09, §6] that for every complete $L_{\omega_1,\omega}$-sentence Φ, there is a complete first-order theory T in a possibly larger countable language such that the reducts of models of Φ are in 1-1 correspondence with the atomic models of T.

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We wish to develop a classification theory for atomic classes $\mathbf{At_T}$ akin to the work of the third author concerning $Mod(T)$ for complete, first order T. In the first-order context, a fundamental dividing line is *superstability*. The third author proved that if T is unsuperstable, then $Mod(T)$ contains 2^κ non-isomorphic models of size κ for each uncountable cardinal κ. On the other hand, if T is superstable, then models of T admit a desirable independence relation, non-forking. From this, one can measure the forking complexity of a type by assigning ranks to the space of types, e.g., $R^\infty(p)$ or $U(p)$. These ranks allow one to prove structural results for $Mod(T)$ by way of inductive arguments on the space of types.

When one is only considering the class $\mathbf{At_T}$ of atomic models of T, the usual dividing lines are not relevant and the test questions need to be altered. It is notable that in the context of atomic models, even (first order) stability is not relevant. See, for example, Example 2.4. There, $T = Th(N)$ is not stable in the first order context, yet $\mathbf{At_T}$ has a unique atomic model in every infinite cardinality. Worse, as the Upward Löwenheim-Skolem theorem can fail for atomic models, asking for many atomic models in all uncountable cardinalities may well be meaningless; e.g. if there are models only up to $\aleph_1$. However, by classical results of Vaught, an atomic model of size $\aleph_1$ exists if and only if the (unique) countable atomic model is not minimal. Thus, it is natural to call an atomic class $\mathbf{At_T}$ *unstructured* if it contains $2^{\aleph_1}$ non-isomorphic atomic models, each of size $\aleph_1$, and then to ask what effect does 'structured' have on its countable models.

In this paper, we introduce and develop a rank, $\mathrm{rk}(d/a)$, on all finite tuples d, a from a fixed countable atomic model N and with Theorem 5.4.2 we prove that if if $\mathbf{At_T}$ is structured (fewer than $2^{\aleph_1}$ non-isomorphic atomic models of size $\aleph_1$) then $\mathrm{rk}(d/a)$ exists for all $d, a \subseteq N$. Our rank rk is similar to U-rank, which, in the first order context, is the foundation rank on the space of complete types, which are tree ordered by the relation $p < q$ iff q is a forking extension of p. In first order, a theory is superstable if and only if $U(p)$ is ordinal valued for every complete type p.

Our rank can also be viewed as a foundation rank of types $\mathrm{tp}(d/a)$ with respect to the relation of 'an extension making some element pseudo-algebraic.' However, because $\mathrm{rk}(d/a)$ is determined by $\mathrm{tp}(d/a)$, which is generated by a complete formula, we additionally get that our rank is continuous, which is not generally true of U-rank in first order, superstable theories. For such theories, an alternate rank is R^∞-rank, but it is a rank on formulas as opposed to types. Its natural generalization to types, given by $R^\infty(p) = \min\{R^\infty(\theta) : \theta \in p\}$ is only semi-continuous. It is pleasing that our rank possesses both of the desirable properties of R^∞-rank and U-rank – it is a rank on types that is fully continuous and is a foundation rank of a natural extendibility property.

The new rank is defined in Section 2, with the salient features developed in Sections 3 and 4. The main results are stated explicitly in Theorem 2.5, but here is a summary. We prove semantic equivalents of the rank in terms of the existence of certain chains, which shows that $\mathrm{rk}(d/a) = 0$ and $\mathrm{rk}(d/a) = 1$ are equivalent to being pseudo-algebraic and pseudo-minimal, respectively. Continuing upward, we see that 'having rank $n < \omega$ is extendible,' i.e., if $\mathrm{rk}(d/a) = n$, then for every type $q \in S_{at}(a)$ there is some b realizing q with $\mathrm{rk}(d/ab) = n$. This result implies that among finitely ranked atomic classes, the rank is fully additive. Using this additivity, we conclude that any finite set is dominated in some sense by an independent sequence of psuedo-minimal types. Collectively, these results show that finitely ranked atomic classes $\mathbf{At_T}$ are similar to finitely ranked superstable theories in the first order context. Finally, in Section 5, we prove our main result, Theorem 5.4.2. Its rather lengthy argument shows that the assumption of 'few atomic models in $\aleph_1$' implies that the class $\mathbf{At_T}$ is ranked. Even though the theorem is proved in ZFC, heavy use is made of certain forcing constructions.

1 Context

En route to proving a first order theory T is categorical in $\aleph_1$ is categorical in all uncountable cardinalities, Morley [Mor65], Morley exploited the upwards Löwenheim-Skolem theorem. He applied the Erdös-Rado theorem to deduce that the unique model in $\aleph_1$ can be represented as an Ehrenfeucht-Mostowski over the cardinal $\aleph_1$ and concluded that the theory admitted only countably many types over a countable set. He called this property *totally transcendental*, now usually called ω-stable. Shelah later extended this result under the set theoretic hypothesis $2^{\aleph_n} < 2^{\aleph_{n+1}}$, by showing a complete $L_{\omega_1,\omega}$-sentence that is categorical in all infinite cardinals below $\aleph_\omega$ is *excellent* and consequently, has arbitrarily large models and is categorical in all uncountable cardinalities.

As described in [BLS16], [She83], and [Bal09, Chapter 6], the models of such a sentence can be thought of as the atomic models (all finite sequences in each model realize a principal type) of a first order theory; we work here with that assumption.

[She83, She09] proved (See also [Bal09, Chapter 17]):

Fact 1.1 (Martin's Axiom)**.** *There is a sentence ψ in $L(Q)$ with the joint embedding property that is κ-categorical for every $\kappa < 2^{\aleph_0}$. In ZFC one can prove ψ is $\aleph_0$-categorical but the associated AEC has neither the amalgamation property in $\aleph_0$ nor is ω-stable.*

$L(Q)$ is first order logic extended by the quantifier, 'there exists countably many'. Shelah proposed a variant to get such a counterexample in $L_{\omega_1,\omega}$; however, there was a gap. This article is part of a more than 20 year effort to fill the gap or, mostly, to prove no such example exists by showing $\aleph_1$-categoricity implies ω-stability; the existence of a model in $\beth_1^+$ is state of the art in that direction [BLS24]. This particular paper is in the midst of those alternatives: establishing the existence of a rank analogous to the U-rank in superstable theories from the assumption that there are few (atomic) models in $\aleph_1$.

The hope was to show that any ranked sentence has a model in the continuum, but sadly that goal remains unattained.

2 The new rank and statements of the main results

Throughout this section, fix a complete theory T in a countable language and assume there is a countable atomic model N that is not minimal. So long as we restrict to complete types of finite tuples over finite subsets of N, N serves as a 'monster model'. For every finite a, N realizes all types over a, and moreover N is homogeneous – if a, b, c are finite tuples from N with $\mathrm{tp}(a/c) = \mathrm{tp}(b/c)$, then there is an automorphism $\sigma \in Aut(N)$ fixing c pointwise with $\sigma(a) = b$.

Throughout Sections 2-4, we assume all finite tuples are from this countable, atomic model N.

We recall a definition from [BLS16, §2].

Definition 2.1. We say d *is in the pseudo-closure of* a, $d \in \mathrm{pcl}(a)$, if every model $M \preceq N$ containing a also contains d. For a finite tuple a, $\mathrm{pcl}(a) = \{d \in N : d \in \mathrm{pcl}(a)\}$ and for $A \subseteq N$ any set, $\mathrm{pcl}(A) = \bigcup\{\mathrm{pcl}(a) : a \in A \text{ finite}\}$.

A complete type $p \in S(a)$ is *pseudo-algebraic* if $d \in \mathrm{pcl}(a)$ for some (equivalently every) d realizing p.

As we are assuming N has a proper elementary substructure, psuedo-closure is not degenerate, i.e., $N \neq \mathrm{pcl}(\emptyset)$.

Definition 2.2. In [BLS16], a type $p = \mathrm{tp}(d/a)$ is *pseudo-minimal* if $d \notin \mathrm{pcl}(a)$ and for every b, c, if $c \in \mathrm{pcl}(abd) \setminus \mathrm{pcl}(ab)$, then $d \in \mathrm{pcl}(abc)$.

We say that the *pseudo-minimal types are dense* if, for every non-pseudo-algebraic $p = \mathrm{tp}(d/a)$, there is some a^* such that $\mathrm{tp}(d/aa^*)$ is pseudo-minimal.

Definition 2.3. Let $\mathbf{P}$ denote the set of all types $\mathrm{tp}(d/a)$ for finite tuples $a, d \subseteq N$. As N is atomic, every $p \in \mathbf{P}$ is principal.

We define a rank $\mathrm{rk} : \mathbf{P} \to \mathbf{ON} \cup \{\infty\}$ by induction on α requiring for any finite sequence a :

- $\mathrm{rk}(d/a) \geq 0$
- $\mathrm{rk}(d/a) \geq \alpha > 0$ if and only if for every $r(y) \in S_{at}(a)$ and for every $\beta < \alpha$, there exist tuples a', b, c from N such that

1. $\mathrm{tp}(a'/a) = r$;
2. $\mathrm{rk}(\mathrm{tp}(d/aa'bc)) \geq \beta$; and
3. $c \in \mathrm{pcl}(daa'b) \setminus \mathrm{pcl}(aa'b)$.

- For an ordinal α, $\mathrm{rk}(p) = \alpha$ if $\mathrm{rk}(p) \geq \alpha$, but $\mathrm{rk}(p) \not\geq \alpha + 1$.
- Call $\mathbf{At_T}$ *ranked* if $\mathrm{rk}(p)$ is ordinal-valued for every $p \in \mathbf{P}$.
- Call $\mathbf{At_T}$ *finitely ranked* if $\mathrm{rk}(p) < \omega$ for every $p \in \mathbf{P}$.

Observe that $d \in \mathrm{pcl}(a)$ if and only if $\mathrm{rk}(d/a) = 0$. While the ranks are on *complete* formulas, not all formulas have been ranked. To remedy this we can define $\mathrm{rk}(\varphi(x, a)) = \sup\{\mathrm{rk}(p) : \varphi(x, a) \in p \in \mathbf{P}\}$. The following example may give some intuition.

Example 2.4. Let $L = \{A, B, \pi, \leq\}$ and let N be the L-structure where A and B partition the universe with B infinite, $\pi : A \to B$ is a total surjective function and $(\pi^{-1}(b), \leq) \cong (\mathbb{Z}, \leq)$, with $a \not\leq a'$ whenever $\pi(a) \neq \pi(a')$. Then N is an atomic model of $T = Th(N)$, and any $M \models T$ will be atomic if and only if $(\pi^{-1}(b), \leq) \cong (\mathbb{Z}, \leq)$ for every $b \in B$.

Now choose elements $a, b \in N$ such that $\pi(a) = b$. Clearly, a is not algebraic over b in the classical sense, however they are "equi-pseudo-algebraic" i.e., $b \in \mathrm{pcl}(a)$ (trivially) and $a \in \mathrm{pcl}(b)$.

In terms of ranks, note that $\mathrm{pcl}(\emptyset) = \emptyset$, so for any $e \in N$, $\mathrm{rk}(e/\emptyset) \geq 1$. For a, b with $\pi(a) = b$, $rk(a/\emptyset) = \mathrm{rk}(b/\emptyset) = 1$ and both of these types are pseudo-minimal. However, $\mathrm{rk}(a/b) = \mathrm{rk}(b/a) = 0$. Here, $\mathbf{At_T}$ is categorical in every infinite power and is finitely ranked. To give an example of a structure of rank 2, add an equivalence relation E and insist that each class is a model of the current example.

We will prove four main results about this rank:

Theorem 2.5. *Let T be a complete theory in a countable language for which there is an uncountable atomic model. Then:*

1. *(Proposition 3.16) For $p = \mathrm{tp}(d/a) \in \mathbf{P}$, $\mathrm{rk}(p) = \infty$ if and only if there is an infinite sequence of models $M_0 \preceq M_1 \preceq \cdots \preceq N$ with $a \in M_0$ and tuples $c_n \in M_{n+1}$ such that $c_n \in \mathrm{pcl}(M_n d) \setminus M_n$ for each n.*
2. *(Theorem 5.4.2) If $\mathbf{At_T}$ has $< 2^{\aleph_1}$ non-isomorphic atomic models of size $\aleph_1$, then $\mathbf{At_T}$ is ranked.*
3. *If $\mathbf{At_T}$ is ranked, then: (Corollaries 3.8,3.9, and Proposition 3.11)*
   - *(a) The pseudo-minimal types are dense;*

(b) If $\mathrm{rk}(d/a) = n < \omega$ *then there is a model* $M \supseteq a$ *with* $\mathrm{rk}(d/M) = n$*;*

(c) Among types of finite rank, the rank is fully additive, i.e., $\mathrm{rk}(de/a) = \mathrm{rk}(d/ea) + \mathrm{rk}(e/a)$ *whenever* $\mathrm{rk}(de/a) < \omega$

4. *(Proposition 4.4) If* $\mathbf{At_T}$ *is finitely ranked, then for every pseudo-minimal* $\theta(x)$*, for every independent tuple* $\overline{c} \in \theta(N)^n$ *and every finite* $b \subseteq N$*, there is a finite* $h \subseteq N$ *for which* $\overline{c}$ θ*-dominates* b *over* h *(Definition 4.3).*

Note that the main result from [BLS16], that failure of density of pseudo-minimal types implies the existence of $2^{\aleph_1}$ non-isomorphic atomic models of size $\aleph_1$, follows immediately from Theorem 2.5.

3 Properties of the rank, chains, and additivity

We first record some properties of the new rank. We begin with two easy monotonicity results.

Lemma 3.1. *Suppose* $d_0 \subseteq d$ *and* $a \subseteq a^*$ *are from* N*. Then:*

1. $\mathrm{rk}(d_0/a) \leq \mathrm{rk}(d/a)$*; and*
2. $\mathrm{rk}(d/a^*) \leq \mathrm{rk}(d/a)$.

Proof. (1) is easy and is left to the reader. For (2), we prove that for any ordinal α, $\mathrm{rk}(d/a^*) \geq \alpha + 1$ implies $\mathrm{rk}(d/a) \geq \alpha + 1$, which suffices. Suppose $\mathrm{rk}(d/a^*) \geq \alpha + 1$. Choose any $r \in S_{at}(a)$ and any realization a_0 of r. Let $r^* := \mathrm{tp}(a_0/a^*)$. As $\mathrm{rk}(d/a^*) \geq \alpha + 1$, choose a realization a' of r^* and b, c such that $\mathrm{rk}(d/a^*a'bc) \geq \alpha$, $c \in \mathrm{pcl}(a^*a'bd) \setminus \mathrm{pcl}(a^*a'b)$. Put $b' := a^*b$. As $a'b' = a'a^*b$ (as sets), a', b', c satisfy $a' \models r$, $\mathrm{rk}(d/a'bc) \geq \alpha$, and $c \in \mathrm{pcl}(a'b'd) \setminus \mathrm{pcl}(a'b')$. Thus, $\mathrm{rk}(d/a) \geq \alpha + 1$. □

The following two Lemmas establish that the rank is continuous.

Lemma 3.2. *For any ordinal* α*, if* $\mathrm{rk}(d/a) = \alpha$*, then for all* $\beta < \alpha$ *there is* β^**,* $\beta \leq \beta^* < \alpha$ *and* $e \subseteq N$ *such that* $\mathrm{rk}(d/ae) = \beta^*$*.*

Proof. First, since $\mathrm{rk}(d/a) \ngeq \alpha + 1$, choose some $r \in S_{at}(a)$ for which there do not exist a', b, c from N such that a' realizes r, $\mathrm{rk}(d/aa'bc) \geq \alpha$ and $c \in \mathrm{pcl}(aa'bd) \setminus \mathrm{pcl}(aa'b)$. Now choose $\beta < \alpha$. By the definition of $\mathrm{rk}(d/a) \geq \alpha$ applied to β and the r from above, there are a', b, c such that a' realizes r, $c \in \mathrm{pcl}(aa'bd) \setminus \mathrm{pcl}(aa'b)$, and $\mathrm{rk}(d/aa'bc) \geq \beta$. Take $e := a'bc$ and $\beta^* := \mathrm{rk}(d/ae)$. □

Lemma 3.3. *For any ordinal α, if $\mathrm{rk}(d/a) = \alpha$, then for every $\gamma < \alpha$ there is some $b \subseteq N$ such that $\mathrm{rk}(d/ab) = \gamma$.*

Proof. We prove this by induction on α. For $\alpha = 0$ there is nothing to prove, so fix $\alpha > 0$ and assume the statement holds for all $\beta < \alpha$. Choose any $\gamma < \alpha$ and apply Lemma 3.2 to get $e \subseteq N$ such that $\mathrm{rk}(d/ae) \in [\gamma, \alpha)$. If $\mathrm{rk}(d/ae) = \gamma$ we are done. Otherwise, apply the inductive hypothesis to $\mathrm{rk}(d/ae)$ to get b^* such that $\mathrm{rk}(d/aeb^*) = \gamma$. Then $b := eb^*$ is as required. □

Proposition 3.4. *1. There is some countable ordinal γ^* such that, for all finite $d, a \subseteq N$, if $\mathrm{rk}(d/a) \geq \gamma^*$, then $\mathrm{rk}(d/a) = \infty$. If $\mathrm{rk}(d/a) \geq \omega_1$, then $\mathrm{rk}(d/a) = \infty$.*

2. If $\mathrm{rk}(d/a) = \infty$, then

(a) For any $r \in S_{at}(a)$ there is a' realizing r with $\mathrm{rk}(d/aa') = \infty$; and

(b) There are $b^*, c \subseteq N$ with $c \in \mathrm{pcl}(ab^*d) \setminus \mathrm{pcl}(ab^*)$ with $\mathrm{rk}(d/ab^*c) = \infty$.

Proof. (1) Let I be the image of the rank function $\mathrm{rk} : \mathbf{P} \to ON \cup \{\infty\}$. By Lemma 3.3 $I \setminus \{\infty\}$ is downward closed and is countable since $\mathbf{P}$ is. Thus, $I \setminus \{\infty\} = \gamma$ for some countable ordinal γ. Then $\gamma^* := \gamma + 1$ satisfies (1).

(2) Suppose $\mathrm{rk}(d/a) = \infty$. Then $\mathrm{rk}(d/a) \geq \gamma^* + 1$ for γ^* as in (1). It follows from the definition of $\mathrm{rk}(d/a)$ that for every $r \in S_{at}(a)$ there are a', b, c in N such that a' realizes r, $c \in \mathrm{pcl}(aa'bd) \setminus \mathrm{pcl}(aa'b)$, and $\mathrm{rk}(d/aa'bc) \geq \gamma^*$. Then $\mathrm{rk}(d/aa') \geq \mathrm{rk}(d/aa'bc) \geq \gamma^*$, so $\mathrm{rk}(d/aa') = \infty$, satisfying (2a). For (2b), take $b^* := a'b$. □

3.1 Finite and infinite chains

We begin with a notational extension of our rank function and show that $\mathrm{rk}(d/a) \geq n$ can be characterized by a certain finite chain of submodels of N.

Definition 3.5. Suppose $A \subseteq N$ is infinite. For $d \subseteq N$ finite, say $\mathrm{rk}(d/A) := \min\{\mathrm{rk}(d/a) : a \subseteq A \text{ finite}\}$. In particular, $\mathrm{rk}(d/A) = \infty$ if and only if $\mathrm{rk}(d/a) = \infty$ for all finite $a \subseteq A$.

Proposition 3.6. *For all $n \in \omega$, for all finite $d, a \subseteq N$, $\mathrm{rk}(d/a) \geq n$ if and only if there is a chain $M_0 \preceq M_1 \preceq \cdots \preceq M_n = N$ with $a \subseteq M_0$ and $\mathrm{pcl}(M_i d) \cap (M_{i+1} \setminus M_i) \neq \emptyset$ for all $i < n$.*

Proof. By induction on n. This is trivial when $n = 0$, so assume the result for n. First, assume $\mathrm{rk}(d/a) \geq n + 1$. Take $r(y) :=$ '$y = a$', $\beta = n$, and choose a', b, c such that a' realizes r (hence $a' = a$), $\mathrm{rk}(d/abc) \geq n$, $c \in \mathrm{pcl}(abd) \setminus \mathrm{pcl}(ab)$. Apply the inductive hypothesis to $\mathrm{tp}(d/abc)$ to get $M_0 \preceq \cdots \preceq M_n = N$ with $abc \subseteq M_0$ and $\mathrm{pcl}(M_i d) \cap$

$(M_{i+1} \setminus M_i) \neq \emptyset$ for all $i < n$. Since $c \notin \mathrm{pcl}(ab)$, there is $M_{-1} \preceq M_0$ with $ab \subseteq M_{-1}$, but $c \in M_0 \setminus M_{-1}$. As $c \in \mathrm{pcl}(abd)$, we have $c \in \mathrm{pcl}(M_{-1}d)$, so $M_{-1} \preceq M_0 \preceq \cdots \preceq M_n = N$ is a requisite chain of length $n + 1$.

Conversely, assume a chain $M_0 \preceq \cdots \preceq M_{n+1} = N$ satisfies $a \subseteq M_0$ and $\mathrm{pcl}(M_i d) \cap (M_{i+1} \setminus M_i) \neq \emptyset$ for all $i < n + 1$. To see that $\mathrm{rk}(d/a) \geq n + 1$, choose any $r \in S_{at}(a)$. Choose any $a' \in M_0$ realizing r. From our assumption on the chain, choose $c \in M_1 \setminus M_0$ with $c \in \mathrm{pcl}(M_0 d)$. Choose a finite $b \subseteq M_0$ such that $c \in \mathrm{pcl}(aa'bd)$. As $c \notin M_0$, $c \notin \mathrm{pcl}(aa'b)$. However, $aa'bc \subseteq M_1$ and the n-chain $M_1 \preceq \cdots \preceq M_n = N$ satisfies $\mathrm{pcl}(M_i d) \cap (M_{i+1} \setminus M_i) \neq \emptyset$ for all $1 \leq i < n + 1$. Thus, $\mathrm{rk}(d/aa'bc) \geq n$ by our inductive hypothesis. The above witnesses that $\mathrm{rk}(d/a) \geq n + 1$, so we are done. □

Proposition 3.6 has many corollaries. The first indicates that the adjectives of pseudo-algebraic and pseudo-minimal occur naturally.

Corollary 3.7. *Suppose $d, a \subseteq N$ are finite sets. Then:*

1. $\mathrm{rk}(d/a) = 0$ *if and only if* $\mathrm{tp}(d/a)$ *is pseudo-algebraic; and*
2. $\mathrm{rk}(d/a) = 1$ *if and only if* $\mathrm{tp}(d/a)$ *is pseudo-minimal.*

Proof. (1) Note that $\mathrm{rk}(d/a) = 0$ iff $\mathrm{rk}(d/a) \ngeq 1$ iff there does not exist $M \preceq N$ with $a \subseteq M$, $d \nsubseteq M$ iff $\mathrm{tp}(d/a)$ is pseudo-algebraic.

(2) First, suppose $\mathrm{rk}(d/a) = 1$. By (1), $\mathrm{rk}(d/a)$ is not pseudo-algebraic. To see that $\mathrm{tp}(d/a)$ is pseudo-minimal, choose any $c \in \mathrm{pcl}(da) \setminus \mathrm{pcl}(a)$, and assume by way of contradiction that $d \notin \mathrm{pcl}(ac)$. Since $d \notin \mathrm{pcl}(ac)$, choose $M_1 \preceq N$ with $ac \subseteq d$, but $d \nsubseteq M_1$. Also, since $c \notin \mathrm{pcl}(a)$, there is $M_0 \preceq M_1$ with $a \subseteq M_0$, but $c \nsubseteq M_0$. Then the 2-chain (M_0, M_1, N) witnesses that $\mathrm{rk}(d/a) \geq 2$. Conversely, assume $\mathrm{tp}(d/a)$ is pseudo-minimal and we show there cannot be a 2-chain $M_0 \preceq M_1 \preceq N$ with $a \subseteq M_0$ and $c \in M_1 \setminus M_0$ with $c \in \mathrm{pcl}(M_0 d) \setminus M_0$. If there were, then taking any finite $b \subseteq M_0$ for which $c \in \mathrm{pcl}(bd)$, the elements a, b, c, d contradict the pseudo-minimality of $\mathrm{tp}(d/a)$. □

Corollary 3.8. *If $\mathbf{At_T}$ is ranked, then the pseudo-minimal types are dense.*

Proof. Choose any $d, a \subseteq N$ such that $\mathrm{tp}(d/a)$ is not pseudo-algebraic. Since $\mathrm{rk}(d/a)$ exists, by Corollary 3.7, $\mathrm{rk}(d/a) \geq 1$. By Lemma 3.3 there is $b \subseteq N$ such that $\mathrm{rk}(d/ab) = 1$, hence $\mathrm{tp}(d/ab)$ is pseudo-minimal. □

Finally, we see that finitely ranked types can be extended to types over models with the same rank.

Corollary 3.9. *Suppose* $\mathrm{rk}(d/a) = n < \omega$. *Then:*

1. *For any* $r \in S_{at}(a)$ *there is* a' *realizing* r *such that* $\mathrm{rk}(d/aa') = n$*.*

2. *There is a model* $M \preceq N$ *with* $a \subseteq M$ *and* $\mathrm{rk}(d/M) = n$*.*

Proof. (1) Using Proposition 3.6, choose an n-chain $M_0 \preceq \ldots M_n = N$ with $a \subseteq M_0$ and $\mathrm{pcl}(M_i d) \cap (M_{i+1} \setminus M_i) \neq \emptyset$ for all $i < n$. Choose any $a' \in r(M_0)$. Then the same chain demonstrates that $\mathrm{rk}(d/aa') \geq n$. Hence $\mathrm{rk}(d/aa') = n$ by Lemma 3.1.

(2) follows immediately by iterating (1) ω times. □

3.2 Additivity

Lemma 3.10. *Suppose* X *is finite and* $\mathrm{tp}(d/X) < \omega$*. If* $c \in \mathrm{pcl}(dX) \setminus \mathrm{pcl}(X)$*, then* $\mathrm{rk}(d/Xc) < \mathrm{rk}(d/X)$*.*

Proof. Say $\mathrm{rk}(d/Xc) = n$ and choose an n-chain $M_0 \preceq \ldots M_n = N$ with $Xc \subseteq M_0$ and $\mathrm{pcl}(M_i d) \cap (M_{i+1} \setminus M_i) \neq \emptyset$ for all $i < n$. Since $c \in M_0$ but $c \notin \mathrm{pcl}(X)$, choose $M_{-1} \preceq M_0$ with $X \subseteq M_{-1}$ and $c \in M_0 \setminus M_{-1}$. This gives an $(n+1)$-chain for $\mathrm{tp}(d/X)$, hence $\mathrm{rk}(d/X) \geq n+1$. □

Note that the same result holds when X is infinite as well (so long as the ranks are finite).

Proposition 3.11. *Suppose* $a, d, e \subseteq N$ *are finite.*

1. *If* $\mathrm{rk}(e/a) = k$ *and* $\mathrm{rk}(d/ae) = \ell$ *are both finite, then* $\mathrm{rk}(de/a) = k + \ell$*, so in particular is finite.*

2. *If* $\mathrm{rk}(de/a) < \omega$*, then* $\mathrm{rk}(de/a) = \mathrm{rk}(e/a) + \mathrm{rk}(d/ae)$*.*

Proof. (1) Choose any $n \in \omega$ such that $\mathrm{rk}(de/a) \geq n$ and we argue that $n \leq k + \ell$. To see this, choose an n-chain $M_0 \preceq \cdots \preceq M_n = N$ with $a \subseteq M_0$ and $\mathrm{pcl}(M_i de) \cap (M_{i+1} \setminus M_i) \neq \emptyset$ for all $i < n$. We argue by induction on i that

$$\mathrm{rk}(e/M_{n-i}) + \mathrm{rk}(d/M_{n-i}e) \geq i$$

for all $0 \leq i \leq n$. For $i = 0$ this is obvious, so assume it holds for $i < n$ and we show this for $i+1$. Let $j = n - i - 1$. Choose $c \in (M_{j+1} \setminus M_j) \cap \mathrm{pcl}(de/M_j)$. There are two cases. If $c \in \mathrm{pcl}(eM_j) \setminus M_j$ then by Lemma 3.10, $\mathrm{rk}(e/M_j) > \mathrm{rk}(e/M_{j+1})$. On the other hand, if $c \in \mathrm{pcl}(deM_j) \setminus \mathrm{pcl}(eM_j)$, then $rk(d/eM_j) > \mathrm{rk}(d/eM_{j+1})$. In either case, the sum is incremented by at least one.

However, by Lemma 3.1, $\mathrm{rk}(e/M_0) \leq \mathrm{rk}(e/a) = k$ and $\mathrm{rk}(d/M_0 e) \leq \mathrm{rk}(ae) = \ell$. It follows that $n \leq k + \ell$. In particular, $n := \mathrm{rk}(de/a)$ is finite. To show that $n \geq k + \ell$, it suffices to produce an n-chain for $\mathrm{tp}(de/a)$. For this, since $\mathrm{rk}(d/ea) = \ell$, find $M_0 \preceq \cdots \preceq M_\ell$ with $ea \subseteq M_0$ and $\mathrm{pcl}(M_i d) \cap (M_{i+1} \setminus M_i) \neq \emptyset$ for all $i < \ell$.

Next, fix an isomorphism $f : N \to M_0$ fixing ea pointwise. Since $\mathrm{rk}(e/a) = k$, there exist $M_{-k} \preceq \cdots \preceq M_0$ with $a \subseteq M_{-k}$ and $\mathrm{pcl}(M_i e) \cap (M_{i+1} \setminus M_i) \neq \emptyset$ for all $-k \leq i < 0$. The concatenation of these gives a $(k+\ell)$-chain $M_{-k} \preceq \ldots M_\ell$ witnessing that $\mathrm{rk}(de/a) \geq k + \ell$.

(2) By monotonicity, if $\mathrm{rk}(de/a) < \omega$, then both $\mathrm{rk}(d/ea)$ and $\mathrm{rk}(e/a)$ are finite as well (and, in fact, are at most $\mathrm{rk}(de/a)$). So we are done by (1). □

We record the following immediate corollary (recall Definition 2.3).

Corollary 3.12. *If $\mathrm{At}_\mathbf{T}$ is finitely ranked then*

$$\mathrm{rk}(ab/c) = \mathrm{rk}(a/bc) + \mathrm{rk}(b/c)$$

for all finite a, b, c from N.

Next, we consider infinite chains and see that their existence characterizes $\mathrm{rk}(d/a) = \infty$. The chains defined here are crucial for the construction of $2^{\aleph_1}$ models in $\aleph_1$.

Definition 3.13. Fix $\mathrm{tp}(d/a) \in \mathbf{P}$. A *$d/a$-chain* is an ω-sequence $\langle (M_i, c_i) : i \in \omega \rangle$ with union M^* such that

1. $a \subseteq M_0$ and c_0 is meaningless;
2. $M_0 \preceq M_1 \preceq \ldots M^* \preceq N$ is a nested sequence of (countable atomic) models;
3. $c_{i+1} \in M_{i+1} \setminus M_i$ for every $i \in \omega$; and
4. $c_{i+1} \in \mathrm{pcl}(M_i d)$.

A *better* d/a-chain also satisfies:

5. For every $c \in \mathrm{pcl}(M^* d)$, if $\mathrm{rk}(d/M^* c) = \infty$, then $c \in M^*$; and
6. For every finite $e \in M^*$, every non-pseudoalgebraic 1-type $q \in S_{at}(e)$ is realized in $N \setminus M^*$.

Note that it follows from Clauses (3) and (4) that $d \notin \bigcup_{i \in \omega} M_i$. With Proposition 3.16 we show that having a (better) d/a-chain characterizes $\mathrm{rk}(d/a) = \infty$. We begin by defining finite approximations of a d/a-chain.

Definition 3.14. A d/a-approximation is a sequence $\mathbf{e} = \langle (e_i, c_i) : i \in \lg(\mathbf{e}) \rangle$ where

1. $a \subseteq e_0 \subseteq N$, $c_0 = \emptyset$ and $\lg(\mathbf{e}) < \omega$;
2. $c_{i+1} \in \mathrm{pcl}(e_i, d) \setminus \mathrm{pcl}(e_i)$; and
3. $e_i \cup \{c_{i+1}\} \subseteq e_{i+1}$;
4. $\mathrm{rk}(d/e_{\lg(\mathbf{e})}) = \infty$.

Lemma 3.15. *Given any $n \geq 1$ and any d/a-approximation $\mathbf{e} = \langle (e_i, c_i) : i \in \lg(\mathbf{e}) \rangle$ of length $n = \lg(\mathbf{e})$, there is a sequence $M_0 \preceq \cdots \preceq M_{n-1} \preceq N$ such that for each $i < n$*

- $e_i \subseteq N_i$*; and*
- $c_{i+1} \notin N_i$

Proof. By reverse induction. First, since $c_n \notin \mathrm{pcl}(e_{n-1})$, there is a (countable, atomic) model N_{n-1} that contains e_{n-1} but not c_n. Next, we work inside N_{n-1}. As $c_{n-1} \subseteq e_{n-1}$, $c_{n-1} \subseteq N_{n-1}$. Since $c_{n-1} \notin \mathrm{pcl}(e_{n-2})$, there is N_{n-2}, which we can construct inside of N_{n-1}, that contains e_{n-2} but not c_{n-1}. Now continue. □

The following Proposition is a slight strengthening of Theorem 2.5(1). Better d/a-chains constitute the Data used in proving Theorem 5.4.2.

Proposition 3.16. *The following are equivalent for any* $\mathrm{tp}(d/a) \in \mathbf{P}$.

1. $\mathrm{rk}(d/a) = \infty$*;*
2. *A d/a-chain exists;*
3. *A better d/a-chain exists.*

Proof. $(3) \Rightarrow (2)$ is trivial.

$(2) \Rightarrow (1)$: Fix a d/a-chain $\langle (M_i, c_i) : i \in \omega \rangle$. To show that $\mathrm{rk}(d/a) = \infty$, it suffices to prove that for all ordinals α,

For every finite $e \subseteq \bigcup M_i$, $\mathrm{rk}(d/e) \geq \alpha$.

We establish this by induction on α. Fix α and assume that this holds for every $\beta < \alpha$. Fix any finite $e \subseteq \bigcup M_i$. We directly argue that $\mathrm{rk}(d/e) \geq \alpha$ from the definition of rk. So fix $r \in S_{at}(e)$ and $\beta < \alpha$. Choose n such that $e \subseteq M_n$. Pick any realization a' of r in M_n. Since $c_{n+1} \in \mathrm{pcl}(M_n d)$, we can find a finite $b \subseteq M_n$ such that $c_{n+1} \in \mathrm{pcl}(ea'bd)$. Let $q = \mathrm{tp}(d/ea'bc_{n+1})$. By our inductive hypothesis, $\mathrm{rk}(q) \geq \beta$. Thus, $\mathrm{rk}(d/e) \geq \alpha$ by the definition of rk.

$(1) \to (3)$: Suppose $\mathrm{rk}(d/a) = \infty$. Trivially, $\langle (a, \emptyset) \rangle$ is a d/a-approximation of length 1. We will construct a d/a-chain satisfying Clause (5) of Definition 3.13 in ω steps and then modify it to obtain Clause (6) as well. To get the first part, it suffices to show that any d/a-approximation can be extended in each of three ways.

Extending the sequence

Fix any d/a-approximation $\mathbf{e}$ of length n. In order to get a d/a-approximation of length $(n+1)$ extending $\mathbf{e}$, first note that $\mathrm{rk}(d/e_{n-1}) = \infty$. Thus, taking $r = \emptyset$ (or, if you prefer, let $r(y) :=$ '$y = e'_{n-1}$) there are b, c such that $c \notin \mathrm{pcl}(e_{n-1}b)$, but $c \in \mathrm{pcl}(e_{n-1}bd)$, and $\mathrm{rk}(d/e_{n-1}bc) = \infty$. So, let $e'_{n-1} = e_{n-1}b$ and $e'_n = e_{n-1}bc$ (with $e'_j = e_j$ for all $j < n-1$).

Enlarging e_j one step toward a model

Fix any d/a-approximation $\mathbf{e} = \langle (e_i, c_i) : i \in \lg(\mathbf{e}) \rangle$ of length $n = \lg(\mathbf{e})$. Choose any $j < n$ and fix any consistent formula $\varphi(x, e_j)$. We will produce a larger d/a-approximation $\mathbf{e}'$ where e'_j contains a realization of $\varphi(x, e_j)$. To do this, first choose a sequence of models $N_0 \subseteq N_1 \subseteq \cdots \subseteq N_{n-1}$ as in Lemma 3.15. So, $c_i \subseteq e_i \subseteq N_i$ and $c_{i+1} \in N_i$. Next, choose a^* from N_j realizing $\varphi(x, e_j)$. Let $r^* = \mathrm{tp}(a^*/e_{n-1})$.

Finally, we apply Proposition 3.4(2) to get a', b, c with a' realizing r^* and $\mathrm{rk}(d/e_{n-1}a'bc) = \infty$. Let $q = \mathrm{tp}(d/e_{n-1}a')$ be the restriction, which also has $\mathrm{rk}(q) = \infty$. Now, for $i < j$, let $e'_i = e_i$, while $e'_i = e_i a'$ for all $j \leq i < n$.

One step toward Clause (5) of Definition 3.13

Fix any d/a-approximation $\mathbf{e} = \langle (e_i, c_i) : i \in \lg(\mathbf{e}) \rangle$ of length $n = \lg(\mathbf{e})$, and choose any $c \in \mathrm{pcl}(e_{n-1}d)$. If $\mathrm{rk}(d/e_{n-1}c) < \infty$, then do nothing at this stage. But if $\mathrm{rk}(d/e_{n-1}c) = \infty$, then affix c to e_{n-1} and continue.

By dovetailing these three processes, we can construct a d/a-chain $\langle (M_n, c_n) : n \in \omega \rangle$ satisfying Clause (5) in ω steps. To obtain Clause (6), Let $M^* = \bigcup\{M_n : n \in \omega\}$. As M^* is countable and $S_{at}(b)$ is countable for every finite tuple b from M^*, there is a countable, atomic model $N' \succeq N$ such that for every finite $b \in M^*$ every non-pseudoalgebraic $q \in S_{at}(b)$ is realized in $N' \setminus M^*$. As $\mathrm{pcl}(Z, N) = \mathrm{pcl}(Z, N')$ for all sets $Z \subseteq N$, we see that $\langle M_n : n \in \omega \rangle$ and M^* is also a d/a-chain satisfying (5) with respect to N' as well as with respect to N. Thus, if $f : N' \to N$ is any isomorphism fixing da pointwise, then $\langle f(M_n) : n \in \omega \rangle$ and $f(M^*)$ are a better d/a-chain in N, as required. $\square$

4 Pseudo-minimal types and finitely ranked classes

Fix a (complete) pseudo-minimal type $p \in S_{at}(\emptyset)$ and assume $\theta(x)$ isolates p. Then for any finite tuple a from N, the relation of pseudo-closure over a, is an exchange space on $\theta(N)$. That is, $(\theta(N), \mathrm{pcl}_a)$, where $c \in \mathrm{pcl}_a(B)$ iff $c \in \mathrm{pcl}(Ba)$ satisfies the van der Waerden axioms. This implies there is a good notion of independence, i.e., for any a and any pcl_a-closed C, any two maximal pcl_a-independent subsets of C have the same cardinality, which we dub the *dimension* of C over a. We note the following easy facts about independent tuples from $\theta(N)$.

Lemma 4.1. *Suppose $\theta(x)$ is a complete, pseudo-minimal formula and $a \subseteq N$ is any finite tuple.*

1. *If a finite tuple $\overline{c} \subseteq \theta(N)$ is independent over a, then there is $M \preceq N$, with $a \subseteq M$, but $\overline{c} \cap M = \emptyset$.*

2. *Say $\overline{c} = \overline{c}_1 \cap \overline{c}_2$ is any partition, then $\overline{c}_1$ is independent over $\overline{c}_2 b$, hence there is a model $M \preceq N$ with $\overline{c}_2 a \subseteq M$ and $\overline{c}_1 \cap M = \emptyset$.*

3. *For any $\overline{c} \in \theta(N)^n$, $\overline{c}$ is independent over a if and only if $\mathrm{rk}(\overline{c}/a) = n$.*

The reader is cautioned, however, that although any two elements of $\theta(N)$ have the same 1-type over the empty set, there can be infinitely many 2-types of independent tuples in $\theta(N)^2$, e.g., if $\mathbf{At_T}$ is the class of atomic models of REF(bin), the theory of infinitely many refining equivalence relations, where each E_{n+1} partitions each E_n-class into two pieces. Thus, the apt analogue of $\theta(N)$ is that of a weakly minimal formula in the first order context. Despite this, we have the following, which follows from the homogeneity of N.

Lemma 4.2. *Suppose $a, b \subseteq N$ are finite and $\overline{c} \in \theta(N)^n$ is independent over a. Then there is $\overline{c}' \in \theta(N)^n$ such that $\mathrm{tp}(\overline{c}/a) = \mathrm{tp}(\overline{c}'/a)$ and $\overline{c}'$ is independent over ab.*

Proof. By Lemma 4.1(1), choose $M \preceq N$ with $a \subseteq M$ and $\overline{c} \cap M = \emptyset$. Choose an isomorphism $f : N \to M$ with $f(a) = a$ and let $b' \in M$ be such that $\mathrm{tp}(ab) = \mathrm{tp}(ab')$. Since $\overline{c} \cap M = \emptyset$, $\overline{c}$ is independent over ab'. Choose an automorphism $\sigma \in Aut(N)$ with $\sigma(a) = a$ and $\sigma(b') = b$. Then $\overline{c}' := \sigma(\overline{c})$ is independent over ab and $\mathrm{tp}(\overline{c}'/a) = \mathrm{tp}(\overline{c}/a)$. $\square$

For the remainder of this section, we assume that $\mathbf{At_T}$ is finitely ranked, so we have full additivity of rank.

Definition 4.3. Suppose $b, h \subseteq N$ and $\overline{c} \subseteq \theta(N)$. We say $\overline{c}$ *θ-dominates b over h* if

1. $\overline{c}$ is independent over h; and

2. For all $\overline{c}^* \subseteq \theta(N)$ and all $h^* \supseteq h$, if $\overline{cc}^*$ is independent over h^*, then $\overline{c}^*$ is independent over $h^*\overline{c}b$.

Under the assumption that $\mathbf{At_T}$ is finitely ranked, the following existence lemma shows that dominating sets are easily attained.

Proposition 4.4. *($\mathbf{At_T}$ finitely ranked). Suppose $b, d \subseteq N$ are finite and $\overline{c} \subseteq \theta(N)$ is independent over d. Then:*

1. *There is a finite $h \supseteq d$ such that $\overline{c}$ θ-dominates b over h.*
2. *Moreover, if $\overline{c}^* \subseteq \theta(N)$ is initially chosen such that $\overline{c}^*\overline{c}$ is independent over d, then we may additionally have $\overline{c}^*\overline{c}$ is independent over h.*

Proof. (1) Among all finite tuples $h \subseteq N$ with $h \supseteq d$ and $\overline{c}$ independent over h, choose one such that $\mathrm{rk}(b/h\overline{c})$ is minimized. We argue that $\overline{c}$ θ-dominates b over h. To see this, choose any $\overline{c}' \subseteq \theta(N)$ and $h' \supseteq h$ such that $\overline{c}'\overline{c}$ is independent over h'. We verify that $\overline{c}'$ is independent over $h'\overline{c}b$ by proving that $\mathrm{rk}(\overline{c}'/h'\overline{c}b) = \mathrm{rk}(\overline{c}'/h'\overline{c}) = \lg(\overline{c}')$. The second equality is clear, as $\overline{c}'\overline{c}$ is independent over h', but the first equality takes some work.

Note that $\overline{c}$ is independent over $h'\overline{c}'$, so the minimality property of h yields

$$\mathrm{rk}(b/h'\overline{c}'\overline{c}) = \mathrm{rk}(b/h'\overline{c})$$

Now we use additivity of rk, Corollary 3.12, twice. On one hand,

$$\mathrm{rk}(b\overline{c}'/h'\overline{c}) = \mathrm{rk}(b/h'\overline{c}\overline{c}') + \mathrm{rk}(\overline{c}'/h'\overline{c})$$

while on the other hand,

$$\mathrm{rk}(b\overline{c}'/h'\overline{c}) = \mathrm{rk}(\overline{c}'/h'\overline{c}b) + \mathrm{rk}(b/h'\overline{c})$$

Combining these three equalities gives the requisite $\mathrm{rk}(\overline{c}'/h'\overline{c}b) = \mathrm{rk}(\overline{c}'/h'\overline{c})$.

(2) Now suppose $\overline{c}^*$ is given in advance with $\overline{c}^*\overline{c}$ independent over d. By (1), choose $h \supseteq d$ with $\overline{c}$ θ-dominating b over h. This h might not have $\overline{c}^*\overline{c}$ independent over h, but we apply Lemma 4.2 to get one that is. Choose $\overline{c}'' \subseteq \theta(N)$ such that $\mathrm{tp}(\overline{c}''/bd\overline{c}) = \mathrm{tp}(\overline{c}^*/bd\overline{c})$ with $\overline{c}''$ independent over $bd\overline{c}h$. Note that since $\overline{c}$ is independent over h, so is $\overline{c}''\overline{c}$.

As they have the same type, choose an automorphism $\sigma \in Aut(N)$ such that $\sigma\restriction_{bd\overline{c}} = id$ and $\sigma(\overline{c}'') = \overline{c}^*$. Put $h^* := \sigma(h)$. By the automorphism, $\overline{c}^*\overline{c}$ is independent over h^*. As well, $h^* \supseteq d$ and since $\mathrm{tp}(\overline{c}bh) = \mathrm{tp}(\overline{c}bh^*)$, $\overline{c}$ θ-dominates b over h^*. □

We obtain the following Corollary that may be helpful in constructing an atomic model of size continuum.

Corollary 4.5. *($\mathbf{At_T}$ finitely ranked) Suppose $\overline{c}^* \subseteq \theta(N)$ is given, and $\overline{c}, b, h, \overline{c}', b'$ satisfy*

1. $\overline{c}$ θ*-dominates* b *over* h*;*

2. $\overline{c}^*\overline{c}$ *is independent over* h*;*

3. $\overline{c}'$ *is independent over* $\overline{c}^* h\overline{c}b$*; and*

4. $\mathrm{tp}(\overline{c}'b'/h) = \mathrm{tp}(\overline{c}b/h)$.

Then $\overline{c}^*$ *is independent over* $h\overline{c}b\overline{c}'b'$*.*

Proof. By (3), $\overline{c}'$ is independent over $\overline{c}^*\overline{c}h$, hence by (2),

$\overline{c}^*\overline{c}'\overline{c}$ is independent over h

so by (1), $\overline{c}^*\overline{c}'$ is independent over $h\overline{c}b$.

As they have the same type, (4) implies that $\overline{c}'$ θ-dominates b' over h. Apply this with $h^* := h\overline{c}b$ givens $\overline{c}^*$ independent over $h\overline{c}b\overline{c}'b'$. □

5 Few atomic models implies ranked

Throughout this section, we assume that $\mathbf{At_T}$ is not ranked, i.e., $\mathrm{rk}(p) = \infty$ for some $p \in \mathbf{P}$. The goal of this section is to prove that $\mathbf{At_T}$ contains a family of $2^{\aleph_1}$ non-isomorphic (atomic) models, each of size $\aleph_1$. To begin, we fix the following Data that is obtained from the existence of a better a^*/a chain via Proposition 3.16. (Note that as we are producing so many models, we may absorb a into the signature so we are left with a better $a^*/\emptyset$ chain.) Similarly, we can always add a constant symbol to the language, thereby assuring that $\mathrm{pcl}(\emptyset) \neq \emptyset$.

Data 5.0.1. *Fix a countable* $N^* \in \mathbf{At_T}$*, an elementary chain* $\langle M_n : n \in \omega\rangle$ *with union* $M^* \preceq N^*$*, a distinguished element* a^**, elements* $c_m \in M_{m+1} \setminus M_m$ *and finite tuples* $\overline{d}_m$ *from* M_m *such that*

1. *Each* $c_m \in \mathrm{pcl}(a^*\overline{d}_m, N^*)$*;*

2. *For every* $e \in N^* \setminus M^*$*, if* $e \in \mathrm{pcl}(M^*a^*, N^*)$*, then* $\mathrm{rk}(a^*/M^*e) < \infty$*; and*

3. *For every finite* $b \in M^*$*, every non-pseudoalgebraic* $q \in S_{at}(b)$ *is realized in* $N^* \setminus M^*$*.*

Our strategy is similar to the proof of the non-structural result in [BLS16]. In Subsection 5.1, which is nearly identical with [BLS16, §4.1], we define a family of orders I^S, indexed by stationary/costationary subsets of ω_1 and discuss weakly striated models indexed by such an I^S. The forcing $(\mathbb{Q}_I, \leq)$ is defined in Subsection 5.2. In Subsection 5.3 we prove Proposition 5.2.2, that verifies that $(\mathbb{Q}_I, \leq)$ forces the existence of atomic N_I with cardinality $\aleph_1$ determined by the order I. Finally, in Subsection 5.4, we show that having this uniform process of forcing an extension allows us build families of atomic models $(N_S : S \subseteq \omega_1)$ *in* $\mathbb{V}$ (as opposed to in a forcing extension) in such a manner that if $S \triangle S'$ is stationary, then $N_S \not\cong N_{S'}$. Theorem 2.5(2) follows easily from this.

5.1 A class of linear orders and weakly striated models

We begin by describing a class of $\aleph_1$-like linear orders, colored by a unary predicate P and an equivalence relation E with convex classes. A related notion of striation was discussed in [BLS16].

Definition 5.1.1. Let $L_{\rm ord} = \{<, P, E\}$ and let $\mathbf{I}^*$ denote all $L_{\rm ord}$-structures $(I, <, P, E)$ satisfying:

1. $(I, <)$ is an $\aleph_1$-like dense linear order (i.e., $|I| = \aleph_1$, but $pred_I(a)$ is countable for every $a \in I$) with minimum element $\min(I)$;
2. P is a unary predicate;
3. E is an equivalence relation on I with convex classes such that
   (a) If $t = \min(I)$ or if $P(t)$ holds, then $t/E = \{t\}$;
   (b) Otherwise, t/E is (countable) dense linear order without endpoints.
4. The condensation I/E is a dense linear order with minimum element, no maximum element, such that both sets $\{t/E : P(t)\}$ and $\{t/E : \neg P(t)\}$ are dense in it.

We are interested in well-behaved proper initial segments of elements of $\mathbf{I}^*$.

Definition 5.1.2. Fix $(I, <, P, E) \in \mathbf{I}^*$. A proper initial segment $J \subseteq I$ is *endless* if it has no maximum element and is *suitable* if, for every $s \in J$ there is $t \in J$, $t > s$, with $\neg E(s, t)$. Finally, call a suitable J *seamless* if $I \setminus J$ has no minimal E-class.

Clearly, J suitable implies J endless. The following Lemma and Construction are Lemma 4.1.3 and Construction 4.1.4 of [BLS16].

Lemma 5.1.3. *If $(I, <, P, E) \in \mathbf{I}^*$ and $J \subseteq I$ is a seamless proper initial segment, then for every finite $\mathcal{S} \subseteq I$ and $w \in J$ such that $w > s$ for every $s \in \mathcal{S} \cap J$, there is an automorphism π of $(I, <, P, E)$ that fixes $\mathcal{S}$ pointwise, and $\pi(w) \notin J$.*

Construction 5.1.4. *Let $S \subseteq \omega_1$ with $0 \notin S$. There is $I^S = (I^S, <, P, E) \in \mathbf{I}^*$ that has a continuous, increasing sequence $\langle J_\alpha : \alpha \in \omega_1 \rangle$ of proper initial segments such that:*

1. *If $\alpha \in S$, then $I^S \setminus J_\alpha$ has a minimum element a_α satisfying $P(a_\alpha)$; and*
2. *If $\alpha \notin S$ and $\alpha > 0$, then J_α is seamless.*

Definition 5.1.5. Fix an atomic $N \in \mathbf{At_T}$ and some $I = (I, <, E, P) \in \mathbf{I}^*$. We say N is *weakly striated by I* if there are ω-sequences $\langle \overline{a}_t : t \in I \rangle$ satisfying:

- $N = \bigcup\{\overline{a}_t : t \in I\}$; (As notation, for $t \in I$, $N_{<t} = \bigcup\{\overline{a}_j : j < t\}$.)
- If $t = \min(I)$, then $\overline{a}_t \subseteq \mathrm{pcl}(\emptyset, N)$;
- For $t > \min(I)$, $a_{t,0} \notin \mathrm{pcl}(N_{<t}, N)$;
- For each s such that $\neg P(s)$ and for every $n \in \omega$, $a_{s,n} \in \mathrm{pcl}(N_{<s} \cup \{a_{s,0}\}, N)$.

The final clause of this definition is weaker than the notion of *striation* [BLS16, Definition 4.5] in that it only constrains levels where P fails. However, the definition of the forcing will put a constraint (albeit weaker) on levels for which P holds.

Note: In the definition above, we allow $a_{s,m} = a_{t,n}$ in some cases when $(s, m) \neq (t, n)$. However, if $s < t$, then the element $a_{t,0} \neq a_{s,m}$ for any m.

The idea of our forcing will be to force the existence of a weakly striated atomic model N_I indexed by the linear order $I \in \mathbf{I}^*$ formed from Construction 5.1.4. We will begin with the array $X_I = \{x_{t,n} : t \in I, n \in \omega\}$ of symbols. The forcing will give us a complete type Γ in the variables X_I in which every finite subset realizes a principal type with respect to T. This Γ defines a congruence on X_I, with $x_{t,n} \sim x_{s,m}$ if $\Gamma \vdash x_{t,n} = x_{s,m}$. The universe of N_I will be $X_I/\sim$, with Γ providing interpretations of each symbol of τ. Such an N_I will have a 'built in' continuous sequence $\langle N_\alpha : \alpha \in \omega_1 \rangle$ of countable, elementary substructures, where the universe of N_α will be $\{[x_{t,n}] : t \in J_\alpha, n \in \omega\}$ for some suitable initial segment J_α of I. The idea will be to use the data concerning the pair of models (M^*, N^*) given in Data 5.0.1 to make

$$\{\alpha \in \omega_1 : I \setminus J_\alpha \text{ has a minimum element}\}$$

(infinitarily) definable in a language τ^* that codes all of the data mentioned above.

5.2 The forcing

Fix the Data from Data 5.0.1. Fix a stationary/costationary subset $S \subseteq \omega_1$ and use Construction 5.1.4 to form $I^S = (I, <, E, P) \in \mathbf{I}^*$. We describe three adjectives on a weakly striated model.

Definition 5.2.1. Suppose N is weakly striated by $(I, <, P, E)$, $J \subseteq I$ suitable, and $b \in N \setminus N_J$.

- b rk ∞-*catches* N_J if, $\mathrm{rk}(b/N_J, N) = \infty$ and for every $e \in N$, $e \in \mathrm{pcl}(N_J \cup \{b\}, N) \setminus N_J$ implies $\mathrm{rk}(b/N_J \cup \{e\}, N) < \infty$.
- b *admits a cofinal chain in* N_J if there is a strictly increasing, cofinal sequence $\langle s_n : n \in \omega \rangle$ from J such that for every n, $\mathrm{pcl}(N_{s_n} \cup \{b\}, N) \cap N_{s_{n+1}} \neq N_{s_n}$.
- b *has bounded effect in* N_J if there exists $s^* \in J$ such that $\mathrm{pcl}(N_s \cup \{b\}, N) \cap N_J = N_s$ for every $s > s^*$ with $s \in J$.

Note that any sequence $\langle s_n : n \in \omega \rangle$ witnessing 'b admits a cofinal chain' must also satisfy $s_n/E < s_{n+1}/E$. As well, any infinite subsequence would also be a witness. Clearly, if b has bounded effect in N_J, then b cannot admit a cofinal chain in N_J.

Proposition 5.2.2. *Suppose $N^*, a^*, M^* = \bigcup_{m \in \omega} M_m$, $\overline{d}_m, c_m$ are from Data 5.0.1 and let $I = I^S$ be from Construction 5.1.4 with S stationary/costationary. There is a c.c.c. forcing $\mathbb{Q}_I$ such that in $V[G]$, there is a full, atomic $N_I \models T$ weakly striated by $(I, <)$ such that:*

1. *For every suitable initial segment $J \subseteq I$, $N_J \preceq N_I$;*
2. *If $t \in I$ and $P(t)$ holds, then $a_{t,0}$ rk ∞-catches and admits a cofinal chain in N_t; and*
3. *If $J \subseteq I$ is seamless, then for every $b \in N_I \setminus N_J$, if b rk ∞-catches N_J, then b has bounded effect in N_J.*

Recall that by naming a constant if necessary, we are assuming $\mathrm{pcl}(\emptyset, M) \neq \emptyset$. Fix a specific complete formula $\gamma(y)$ isolating a specific type of an element in $\mathrm{pcl}(\emptyset, M)$. Additionally, fix, for the whole of the proof, some $(I, <, E, P) \in \mathbf{I}^*$. We aim to construct an atomic model $N_I \in \mathbf{At_T}$, whose complete diagram consists of $\{x_{t,n} : t \in I, n \in \omega\}$ that is weakly striated by $(I, <)$. We first accomplish this via the forcing notion $(\mathbb{Q}_I, \leq_{\mathbb{Q}}$), defined below. Elements of $\mathbb{Q}_I$ will record 'finite approximations' of this complete diagram. More precisely:

Definition 5.2.3. An *approximation sequence* $\overline{x}$ *from* $\langle x_{t,n} : t \in I, n \in \omega\rangle$ has the form $\overline{x} = \langle x_{t,m} : t \in u, m < n_t\rangle$, where $u \subseteq I$ is finite and $1 \leq n_t < \omega$ for every $t \in u$. Given a finite sequence $\overline{x}$ indexed by u and $\langle n_t : t \in u\rangle$ and given a proper initial segment $J \subseteq I$, let $u\restriction_J = u \cap J$ and $\overline{x}\restriction_J = \langle x_{t,m} : t \in u\restriction_J, m < n_t\rangle$. As well, if $p(\overline{x})$ is a complete type in the variables $\overline{x}$, then $p\restriction_J$ denotes the restriction of p to $\overline{x}\restriction_J$, which is necessarily a complete type. For $s \in I$, the symbols $u\restriction_{<s}$ and $\overline{x}\restriction_{\leq s}$ are defined analogously, setting $J = I\restriction_{<s}$ and $I\restriction_{\leq s}$, respectively.

Definition of $(\mathbb{Q}_{\mathbf{I}}, \leq_{\mathbb{Q}})$**:** $p \in \mathbb{Q}_I$ if and only if the following conditions hold:

1. p is a complete (principal) type with respect to T in the variables $\overline{x}_p$, which are a finite sequence indexed by $u_p \subset I$ and $n_{p,t} \in \omega$. In addition, p comes equipped with a pair of functions $\overline{g}_p = (g_{0,p}, g_{1,p})$, each of which have domain $u_p \cap P$. . (When p is understood we sometimes write n_t, g_0, g_1, etc.);

2. Striation constraints:

   (a) $x_{\min(I),0} \in \overline{x}_p$ and $\gamma(x_{\min(I),0}) \in p(\overline{x}_p)$;

   (b) If $t = \min(I)$, then p 'says' $\{x_{t,n} : n < n_t\} \subseteq \operatorname{pcl}(\emptyset)$;

   (c) For all $t \in u_p$, $t \neq \min(I)$, p 'says' $x_{t,0} \notin \operatorname{pcl}(\overline{x}_p\restriction_{<t})$; and

   (d) For all $s \in u_p$ such that $\neg P(s)$ and $m < n_s$, p 'says' $x_{s,m} \in \operatorname{pcl}(\overline{x}_p\restriction_{<s} \cup \{x_{s,0}\})$.

3. For each $t \in u_p \cap P$, $g_0(t)$ is a finite approximation to a cofinal sequence below t:

   (a) $\operatorname{dom}(g_0(t))$ is a positive integer $\ell_{p,t}$ and for $i < \ell_{p,t}$ we write s_i for $g_0(t)(i)$;

   (b) Every $s_i \in u_p$ and $\min(I) < s_0/E < s_1/E < \cdots < s_{\ell_{p,t}-1}/E < t$;

   (c) $s_{\ell_{p,t-1}}$ is in the topmost E-class of u_p below t; and

   (d) $\neg P(s_i)$

4. For each $t \in u_p \cap P$, $g_1(t)$ gives an 'elementary map' from $\overline{x}_p\restriction_{\leq t}$ into N^*. For $X \subseteq \overline{x}_p\restriction_{\leq t}$, let $g_1(t)[X] := \{g_1(t)(x) : x \in X\}$, the image of $g_1(t)\restriction_X$. We require that each $g_1(t)$ satisfy:

   (a) For any subset $\overline{w} \subseteq \overline{x}_p\restriction_{\leq t}$, and τ-formula $\varphi(\overline{w})$, $p(\overline{x}_p) \vdash \varphi(\overline{w})$ if and only if $N^* \models \varphi(g_1(t)(\overline{w}))$;

   (b) $g_1(t)(x_{t,0}) = a^*$;

   (c) $g_1(t)[\overline{x}_p\restriction_{=t}] \subseteq N^* \setminus M^*$; and

(d) $g_1(t)[\overline{x}_p\restriction_{<t}] \subseteq M^*$

5. Interconnections: For every $t \in u_p \cap P$ and for each $i < \ell_{p,t-1}$, there is some $m \in \omega$ such that, letting $s_i = g_0(t)(i)$ and recalling Data 5.0.1:

   (a) $\overline{d}_m \subseteq g_1(t)[\overline{x}_p\restriction_{\leq s_i/E}]$;

   (b) $g_1(t)(x_{s_{i+1},0}) = c_m$

For $p, q \in \mathbb{Q}_I$, we define $p \leq_{\mathbb{Q}_I} q$ if and only if $\overline{x}_p \subseteq \overline{x}_q$, the complete type $p(\overline{x}_p)$ is the restriction of $q(\overline{x}_q)$ to $\overline{x}_p$, and for all $t \in u_p \cap P$, $g_{0,q}$ end extends $g_{0,p}$, and $g_{1,q}$ extends $g_{1,p}$.

We make the following observations:

- Striation constraint 2(a) implies that $\min(I) \in u_p$ for every $p \in \mathbb{Q}_I$;
- For any $p \in \mathbb{Q}_I$, for each $t \in u_p \cap P$, because of 3(c), $\neg P(\max(u_p \cap I\restriction_{<t}))$;
- Because of the interplay between 4(c,d), for any $p \in \mathbb{Q}_I$, for any $x_{t,n}, x_{s,m} \in \overline{x}_p$, if $P(t)$ holds and $s < t$, then $p \vdash x_{t,n} \neq x_{s,m}$; and
- If $t, t' \in u_p \cap P$ with $t' \neq t$, then, other than each of $g_1(t)$ and $g_1(t')$ being elementary maps on their common domains, there is no assumption of coherence between these maps. In particular, if $t < t'$, we do not enforce that $g_1(t)[\overline{x}_p\restriction_{<t}]$ be contained in $g_1(t')[\overline{x}_p\restriction_{<t'}]$.

5.3 Proof of Proposition 5.2.2

We begin this rather lengthy subsection by describing ways of extending conditions, The first two Lemmas are immediate.

Lemma 5.3.1. *For every $p \in \mathbb{Q}_I$ and suitable $J \subseteq I$, $p\restriction_J \in \mathbb{Q}_I$ and $p\restriction_J \leq_{\mathbb{Q}_I} p$.*

Lemma 5.3.2. *Every automorphism π of $(I, <, E, P)$ naturally extends to an automorphism π' of $\mathbb{Q}_I$ via the mapping $x_{t,n} \mapsto x_{\pi(t),n}$.*

Our aim is to prove Proposition 5.3.19 below, which yields that the forcing is ccc. This will require a number of preparatory lemmas.

Definition 5.3.3. A condition $p \in \mathbb{Q}_I$ is *non-trivial* if $u_p \neq \{\min(I)\}$. [Recall that $\min(I) \in u_p$ by Striation constraint 1(a).]

Definition 5.3.4. For $p, q \in \mathbb{Q}_I$, we say q *is a simple extension of* p if $q \geq_{\mathbb{Q}} p$ and 'u_q has no new E-classes' i.e., for every $s \in u_q$ there is $s' \in u_p$ such that $E(s, s')$.

What makes simple extensions simple is that (among other things) g_0 cannot be increased.

Lemma 5.3.5. *If q is a simple extension of p, then*

1. $u_q \cap P = u_p \cap P$*; and*
2. $g_{0,q} = g_{0,p}$.

Proof. (1) is immediate, since $P(t)$ implies that $t/E = \{t\}$. (2) First, because of (1), $g_{0,q}$ and $g_{0,p}$ have the same domain. However, for any $t \in u_p \cap P$, the definition of $q \geq_{\mathbb{Q}} p$ implies that $g_{0,q}(t)$ end extends $g_{0,p}(t)$. In addition, the 'last element' $g_{0,q}(t)(\ell_{q,t})$ is an element of the largest E-class below t. But, by simplicity, the largest E-class of p below t is equal to the largest E-class below t. As Condition 3(b) implies that the s_i/E are strictly increasing, we must have $g_{0,q}(t) = g_{0,p}(t)$. $\square_{5.3.5}$

Definition 5.3.6. Suppose that $\varphi(y, \overline{z})$ is a complete formula with respect to T and $p \in \mathbb{Q}_I$. We say $\varphi(y, \overline{z})$ *includes* p if $\overline{z} = \overline{x}_p$ and $\varphi(y, \overline{z}) \vdash p(\overline{x}_p)$. For each $w \in u_p$, define $\varphi_w(y, \overline{x}_p{\restriction}_{\leq w})$ to be the restriction of φ to the displayed variables. Clearly, each such $\varphi_w(y, \overline{x}_p{\restriction}_{\leq w})$ is a complete formula that includes $p{\restriction}_{\leq w}$.

As $p(\overline{x}_p)$ describes a complete type and the relation '$a \in \mathrm{pcl}(\overline{b}, M)$' only depends on the complete type $\mathrm{tp}(a\overline{b}, M)$, we say that a complete formula $\varphi(y, \overline{x}_p)$ that includes p is *pseudo-algebraic* if for some/every $M \in \mathbf{At_T}$ and for some/every $\overline{b}$ from M realizing $p(\overline{x}_p)$, $\varphi(y, \overline{b})$ is pseudo-algebraic in M.

Remark 5.3.7. Note that if $\varphi(y, \overline{x}_p)$ is not pseudo-algebraic and $\theta(\overline{z}, \overline{x}_p)$ is any complete formula, then there is a complete, non-pseudo-algebraic $\psi(y, \overline{z}, \overline{x})$ extending $\varphi(y, \overline{x}_p) \wedge \theta(\overline{z}, \overline{x}_p)$. To see this, choose any $c\overline{a}$ realizing $\varphi(y, \overline{x}_p)$ in N. As $\varphi(y, \overline{a})$ is not pseudo-algebraic, choose $M \preceq N$ containing $\overline{a}$ but not c. Choose any $\overline{e}$ from M witnessing $\theta(\overline{e}, \overline{a})$. Then the complete formula ψ isolating $\mathrm{tp}(c, \overline{e}, \overline{a})$ in N suffices.

Lemma 5.3.8. *Suppose $p \in \mathbb{Q}_I$ is non-trivial and $\varphi(y, \overline{x}_p)$ includes p. Then there is a one-point, simple extension q of p such that:*

1. $\overline{x}_q = \{x_{s,m}\} \cup \overline{x}_p$ *for some* $(s, m) \in I \times \omega$*;*
2. $q(\overline{x}_q) = \varphi(x_{s,m}, \overline{x}_p)$*;*
3. *for every $w \in u_p$, if $\varphi_w(y, \overline{x}_p{\restriction}_{\leq w})$ is pseudo-algebraic, then $s \leq w$; and*

4. *if $\varphi(y, \overline{x}_p)$ is not pseudo-algebraic and $u_p \subseteq J$ for some endless J, then we can require $s \in J$ as well.*

Moreover, given any sequence $\langle (t, c_t) : t \in u_p \cap P, c_t \in N^* \rangle$ such that for each t, $\mathrm{tp}(c_t, g_{1,p}(t)[\overline{x}_p\restriction_{\leq t}], N^*)$ contains $\varphi_t(y, \overline{x}_p\restriction_{\leq t})$, we can find q as above with the additional property that $g_{1,q}(x_{s,m}) = c_t$ for all $t \geq s$.

Proof. Our proof will split into several cases. However, in all cases, as we are requiring q to be a simple extension, we must have $u_q \cap P = u_p \cap P$ and $g_{0,q} = g_{0,p}$. Consequently, since $p \in \mathbb{Q}_I$, Constraint groups 3 and 5 will be automatically satisfied for q.

As p is non-trivial, $\max(u_p) \neq \min(I)$.

Case 1a: $\varphi(y, \overline{x}_p)$ is not pseudo-algebraic and $\neg P(\max(u_p))$.

Let $s' = \max(u_p)$. Given any endless J with $u_p \subseteq J$, as s'/E is dense we can choose $s \in s'/E$ such that $s \in J$, but $s > w$ for all $w \in u_p$. Take $m = 0$. Let $\overline{x}_q = \{x_{s,0}\} \cup \overline{x}_p$ and let $q(\overline{x}_q) = \varphi(x_{s,0}, \overline{x}_p)$. Take $\overline{g}_q = \overline{g}_p$ and there is really nothing to check.

Case 1b: $\varphi(y, \overline{x}_p)$ is not pseudo-algebraic and $P(\max(u_p))$ holds.

Let $t^* = \max(u_p)$. We now consider two Subcases, depending on our choice of sequence $\langle (t, c_t) : t \in u_p \cap P \rangle$ in the 'Moreover clause.'

Subcase: $c_{t^*} \in N^* \setminus M^*$. Here, let $(s, m) = (t^*, n_{t^*})$. Put $\overline{x}_q = \{x_{t^*,n_{t^*}}\} \cup \overline{x}_p$ and $q(\overline{x}_q) = \varphi$. For $t \in u_p \cap P$ with $t < t^*$, put $g_{1,q}(t) = g_{1,p}(t)$ and there is nothing to check. Finally, let $g_{1,q}(t^*)$ be the one-element extension of $g_{1,p}(t^*)$ formed by putting $g_{1,q}(t^*)(n_{t^*}) = c_{t^*}$. As we are adding a new point at a level where P holds, the Striation constraints are trivially satisfied, and everything is easy.

Subcase: $c_{t^*} \in M^*$. Here, Constraint 4(c) forbids us from putting the new element at level t^*. Let $s' = g_{0,p}(t^*)(\ell_{t^*} - 1)$. Then $\neg P(s')$, and s'/E is the maximal E-class represented in u_p below t^*. Choose $s \in I$ such that $E(s, s')$ holds, but $s > w$ for every $w \in u_p \setminus \{t^*\}$. Take $m = 0$. That is, $\overline{x}_q = \{x_{s,0}\} \cup \overline{x}_p$ and put $q(\overline{x}_q) = \varphi$. As above, put $g_{1,q}(t) = g_{1,p}(t)$ for every $t \in u_p \cap P$ with $t < t^*$. Finally, let $g_{1,q}(t^*)$ be the one-point extension of $g_{1,p}(t^*)$ formed by putting $g_{1,q}(t^*)(x_{s,0}) = c_t$.

The non-trivial point to check is that this extension q preserves the Striation constraints. To see this, for $w \in u_p \setminus \{t^*\}$ there is nothing to check. As $\varphi(y, \overline{x}_p)$ is non-pseudo-algebraic, $x_{s,0} \notin \mathrm{pcl}(\overline{x}_q\restriction_{<s})$. And finally, as $g_{1,p}(t^*)(x_{t^*,0}) = a^* \notin M^*$, while $\{c_t\} \cup g_{1,p}(t^*)[\overline{x}_p\restriction_{<t^*}] \subseteq M^*$, we conclude that $x_{t^*,0} \notin \mathrm{pcl}(\overline{x}_q\restriction_{<t^*})$.

For the remainder of the proof, assume that $\varphi(y, \overline{x})$ is pseudo-algebraic. Indeed, let $w^* \in u_p$ be least such that the restriction $\varphi_{w^*}(y, \overline{x}_p\restriction_{\leq w^*})$ is pseudo-algebraic. In each of the cases below, we will either take $s = w^*$, or s will be less than w^* but greater than any

$w \in u_p$ with $w < w^*$. Thus, the Striation constraints for each $w \in u_q$ with $w < s$ will be trivially satisfied since they hold for p. Also, since $\varphi_{w^*}(y, \overline{x}_p\restriction_{\leq w^*})$ is pseudo-algebraic, for every $w \in u_q$ with $w > w^*$ we will have $\mathrm{pcl}(\overline{x}_q\restriction_{\leq w}) = \mathrm{pcl}(\overline{x}_p\restriction_{\leq w})$. It follows from this that the Striation constraints for q are satisfied for every $w \in u_q$ with $w > w^*$. Because of this, in each of the cases below, we only need to establish the Striation constraints for q at levels s and w^*.

Case 2a: $\neg P(w^*)$.

In this case, let $(s, m) = (w^*, n_{p,w^*})$. Take $\overline{x}_q = \{x_{s,m}\} \cup \overline{x}_p$ and $q(\overline{x}_q) = \varphi$. Note that since $\varphi_{w^*}(y, \overline{x}_p\restriction_{\leq w^*})$ is pseudo-algebraic, the Striation constraints are satisfied for q at level w^*, hence at all levels by the comments above. To complete the description of q, for $t < w^*$, put $g_{1,q}(t) = g_{1,p}(t)$. As well, given any sequence $\langle(t, c_t) : t \in u_p \cap P\rangle$ satisfying the 'Moreover clause,' for each $t > w^*$, let $g_{1,q}(t)$ be the one-point extension of $g_{1,p}(t)$ formed by putting $g_{1,q}(t) = c_t$. Constraint group 4 is trivially satisfied for q for each $t < w^*$. So fix $t > w^*$. By Constraint 4(d) on p, we have that $A = g_{1,p}(t)[\overline{x}_p\restriction_{<t}] \subseteq M^*$. However, since $t > w^*$, we also know that the restriction of φ to the variables $(y, \overline{x}_p\restriction_{<t})$ is pseudo-algebraic. Thus, it follows that $c_t \in M^*$. So Constraint 4(d) holds for $g_{1,q}(t)$ as well. The other conditions are trivially satisfied, so $q \in \mathbb{Q}_I$.

Case 2b: $P(w^*)$ holds and, letting $t^* = w^*$, $c_{t^*} \in N^* \setminus M^*$.

In this case, we can place the new element at level t^*. That is, $\overline{x}_q = \{x_{t^*,m}\} \cup \overline{x}_p$, where $m = n_{p,t^*}$ and $q(\overline{x}_q) = \varphi$. As $P(t^*)$ holds, the Striation constraints at level t^* hold for q as they held for p. As noted above, this implies that the Striation constraints hold for q at all levels. Now, for $t < t^*$, let $g_{1,q}(t) = g_{1,p}(t)$ and for $t \geq t^*$, let $g_{1,q}(t)$ be the one-point extension of $g_{1,p}(t)$ formed by putting $g_{1,q}(t)(x_{t^*,m}) = c_t$. We must verify that Constraint group 4 is satisfied. For $t < t^*$, this is trivial. At level t^*, there is no problem as $c_{t^*} \in N^* \setminus M^*$. For levels $t > t^*$, we argue just as in Case 2a) above. That is, since $\varphi_{t^*}(y, \overline{x}_p\restriction_{\leq t^*})$ is pseudo-algebraic, we must have that $c_{t^*} \in M^*$. Thus, there is no problem.

Case 2c: $P(w^*)$ holds and, letting $t^* = w^*$, $c_{t^*} \in M^*$.

As in the second Subcase above, Condition 4(c) forbids us from adding the new element at level t^*. Let $s' \in u_p$ be maximal below t^*. As $\neg P(s')$ holds, s'/E is dense linear order, so we can choose $s \in s'/E$ with $s > s'$. Let $\overline{x}_q = \{x_{s,0}\} \cup \overline{x}_p$ and $q(\overline{x}_q) = \varphi$. We need to check the Striation constraints at levels s and w^*. At level s, note that the minimality of $w^* = t^*$ implies that $x_{s,0} \notin \mathrm{pcl}(\overline{x}_q\restriction_{<s})$, so we are fine at level s. At level $w^* = t^*$, note that $A = g_{1,p}(t^*)[\overline{x}_p\restriction_{<t^*}] \subseteq M^*$ and we are assuming $c_{t^*} \in M^*$. As well, $g_{1,p}(t^*)(x_{t^*,0}) = a^*$, which is in $N^* \setminus M^*$. Thus, the elementarily of the map assumed by the 'Moreover clause' implies that $x_{t^*,0} \notin \mathrm{pcl}(\overline{x}_q\restriction_{<t^*})$, so the Striation constraints are satisfied at level t^* as well.

Finally, we complete the description of q by putting $g_{1,q}(t) = g_{1,p}(t)$ for all $t < t^*$, and for $t \geq t^*$, let $g_{1,q}(t)$ be the one-point extension of $g_{1,p}(t)$ given by $g_{1,q}(t)(x_{s,0}) = c_t$. We need to show that Constraint group 4 is maintained. This is trivial for all $t < t^*$. At level t^*, it is satisfied because of our assumption that $c_{t^*} \in M^*$. Finally, fix any $t > t^*$ and recall that $A = g_{1,p}(t)[\overline{x}_p\restriction_{<t}] \subseteq M^*$. As $x_{s,0} \in \mathrm{pcl}(\overline{x}_p\restriction_{<t})$, this means that $c_t \in \mathrm{pcl}(A, N^*)$, so $c_t \in M^*$, as required in 4(d). $\square_{5.3.8}$

Although Lemma 5.3.8 is very strong, we still need some technique for producing extensions that need not be simple. [Indeed, if p is trivial, then Lemma 5.3.8 does not apply to p at all.]

Lemma 5.3.9. *Suppose $p \in \mathbb{Q}_I$ and $s \in I$ is chosen so that $s' < s$ for every $s' \in u_p$ and $\neg P(s)$. Let $\varphi(y, \overline{x}_p)$ be any complete formula including p that is not pseudo-algebraic. There is $q \in \mathbb{Q}_I$ with $q \geq p$, $\overline{x}_q = \{x_{s,0}\} \cup \overline{x}_p$, and $q(\overline{x}_q) = \varphi$.*

Proof. Simply put $\overline{x}_q = \{x_{s,0}\} \cup \overline{x}_p$ and $q(\overline{x}_q) = \varphi$. As $u_q = u_p \cup \{s\}$ and as $q\restriction_{\leq t} = p\restriction_{\leq t}$ for every $t \in u_p \cap P$, we can put $\overline{g}_q = \overline{g}_p$. That q satisfies the Striation constraints is immediate as φ is not pseudo-algebraic. Thus, $q \in \mathbb{Q}_I$ and $q \geq p$ as required. $\square_{5.3.9}$

Definition 5.3.10. A non-trivial $p \in \mathbb{Q}_I$ is *s-topped* if $\neg P(\max(u_p))$.

Note that for any $p \in \mathbb{Q}_I$ and any $t \in u_p \cap P$, Clauses 3(c),(d) in the definition of the forcing imply that $p\restriction_{<t}$ is s-topped.

Lemma 5.3.11. *Given any $p \in \mathbb{Q}_I$ and any $s' \in I$ such that $u_p \subseteq I\restriction_{<s'}$, there is an s-topped $q \geq p$ with $u_q \subseteq I\restriction_{<s'}$.*

Proof. Choose any $p \in \mathbb{Q}_I$. If p is s-topped, take $q = p$. Otherwise, recall that $\mathrm{tp}(a^*, N^*)$ is not pseudo-algebraic. Let $\delta(y)$ be the complete formula generating $\mathrm{tp}(a^*, N^*)$. By Remark 5.3.7, choose $\varphi(y, \overline{x}_p)$ to be complete, include p, extend $\delta(y)$, but not be pseudo-algebraic. Then apply Lemma 5.3.9 with this φ to get $q \geq p$ as required. $\square_{5.3.11}$

Lemma 5.3.12. *For every $p \in \mathbb{Q}_I$, for every endless $J \supseteq u_p$, for every $t \in u_p \cap P$, and for every finite $C \subseteq N^*$, there is a simple extension q of p satisfying $u_q \subseteq J$ and $C \subseteq range[g_{1,q}(t)]$.*

Proof. Arguing by induction on $|C|$, it suffices to prove this for $C = \{c\}$ a singleton. Fix $p \in \mathbb{Q}_I$, $J \supseteq u_p$ and $t \in u_p \cap P$. As $t \in u_p$, p is non-trivial. Let $B = range[g_{1,p}(t)]$ and, letting $\overline{z} = \overline{x}_p\restriction_{\leq t}$, let $\psi(y, \overline{z}) = \mathrm{tp}(cB, N^*)$. Extend ψ to a complete formula $\varphi(y, \overline{x}_p)$ that includes $p(\overline{x}_p)$. Apply Lemma 5.3.8 to p and φ, using the 'Moreover clause' to require that $g_{1,q}(x_{s,m}) = c$. $\square_{5.3.12}$

Lemma 5.3.13. *If p is s-topped and $\varphi(\overline{y}, \overline{x}_p)$ is any complete formula that includes p, then there is a simple extension q of p such that $q(\overline{x}_q) = \varphi$.*

Proof. Arguing by induction on $\lg(\overline{y})$, we may assume $\overline{y}$ is a singleton. But then, as p s-topped implies p non-trivial, the result follows immediately from Lemma 5.3.8. $\square_{5.3.13}$

Next, we have a series of Lemmas aimed at proving Proposition 5.3.19, which gives a sufficient condition for two conditions to have a common extension. For the proof, we distinguish two cases. The condition is on the sets u_p, u_q, and J, but they collectively describe when we need to increase the sequence described by $g_{0,p}(t^*)$.

Definition 5.3.14. Suppose u, v are finite subsets of I and $J \subseteq I$ is endless. We say v *obstructs u at J* if

1. $t^* = \min(u \setminus J)$ exists and $P(t^*)$ holds;
2. $v \subseteq J$ is non-empty;
3. Taking $s^* = \max(v)$, we have $\neg P(s^*)$, but $s^*/E > s/E$ for every $s \in u \cap J$.

Lemma 5.3.15. *Suppose $p, q \in \mathbb{Q}_I$, J is endless, $u_q \subseteq J$, and $w = \max(u_p \cap J)$. If $E(w, \max(u_q))$, then u_q does not obstruct u_p at J. In particular, if q is a simple extension of $p|J$, then u_q does not obstruct u_p at J.*

Lemmas 5.3.16 and 5.3.17 prove the 'easier half' of Proposition 5.3.19 as we do not need to extend g_0.

Lemma 5.3.16. *Suppose $p, q \in \mathbb{Q}_I$, $J \subseteq I$ is endless, $u_q \subseteq J$, $u_p \setminus J = \{w^*\}$ is a singleton $p\restriction_J \leq q$, and u_q does not obstruct u_p at J. Then there is $r \in \mathbb{Q}_I$ with $\overline{x}_r = \overline{x}_p \cup \overline{x}_q$, $r \geq p$, and $r \geq q$. Moreover, if q is a simple extension of $p\restriction_J$, then r is a simple extension of p.*

Proof. We split into two cases, depending on whether or not $P(w^*)$ holds. In both cases, as the r we construct will satisfy $\overline{x}_r = \overline{x}_p \cup \overline{x}_q$, the primary objective is to find an appropriate complete type $r(\overline{x}_r)$ extending $p(\overline{x}_p) \cup q(\overline{x}_q)$.

Case 1. $\neg P(w^*)$ holds. [Put $s^* = w^*$ to indicate this.]

Write the variables of $\overline{x}_p$ as $y, \overline{y}, \overline{z}$, where $y = x_{s^*,0}$, $\overline{y} = \langle x_{s^*,j} : 1 \leq j < n_{s^*}\rangle$ and $\overline{z} = \overline{x}_p\restriction_{<s^*}$. As $p(y, \overline{y}, \overline{z})$ is a complete type, so is $\varphi(y, \overline{z}) := \exists \overline{y} p(y, \overline{y}, \overline{z})$. By the Striation Constraints, $\varphi(y, \overline{z})$ is not pseudo-algebraic, but $p(\overline{y}; y\overline{z})$ is pseudo-algebraic. Next, write the variables $\overline{x}_q$ as $\overline{z}, \overline{w}$ (this uses $p\restriction_{<s^*} = p\restriction_J$ and $p\restriction_J \leq q$).

Choose any $M \in \mathbf{At_T}$ and choose $AB \subseteq M$ such that $\operatorname{tp}(AB, M) = q(\overline{z}, \overline{w})$. As $\varphi(y, A)$ is not pseudo-algebraic, we can find $c \in M$ such that $M \models \varphi(c, A)$, but $c \notin$

$\mathrm{pcl}(AB, M)$. Then choose $\overline{d}$ from M so that $\mathrm{tp}(c\overline{d}A, M) = p(y, \overline{y}, \overline{z})$. Necessarily, $\overline{d} \subseteq \mathrm{pcl}(cAB, M)$.

Now write $\overline{x}_r$ as $y, \overline{y}, \overline{z}, \overline{w}$ and let $r(\overline{x}_r) = \mathrm{tp}(c\overline{d}AB, M)$. It is easily checked that the Striation Constraints are maintained. As for $\overline{g}_r$, put $\overline{g}_r = \overline{g}_q$, i.e., for every $t \in u_q \cap P$, $g_{0,r}(t) = g_{0,q}(t)$ and $g_{1,r}(t) = g_{1,q}(t)$. As $s^* > s$ for every $s \in u_q$, the functions $\overline{g}_r$ are as required, simply because they were for q.

Case 2. $P(w^*)$ holds. [Put $t^* = w^*$ to indicate this.]

Here, write $\overline{x}_p$ as $\overline{y}, \overline{z}$, where $\overline{y}$ is $\overline{x}_p\restriction_{=t^*}$ and $\overline{z}$ is $\overline{x}_p\restriction_{<t^*}$. As $p\restriction_J = p\restriction_{<t^*}$ and $p|J \leq q$, we can write $\overline{x}_q$ as $\overline{z}, \overline{w}$ and we have $q(\overline{z}, \overline{w}) \vdash p(\overline{z})$. In this case, we use the function $g_{1,p}(t^*) : \overline{yz} \to N^*$ as our guide. We know that $B = g_{1,p}(t^*)[\overline{y}] \subseteq N^* \setminus M^*$, $A = g_{1,p}(t^*)[\overline{z}] \subseteq M^*$, and $\mathrm{tp}(BA, N^*) = p(\overline{y}, \overline{z})$. As $q(\overline{z}, \overline{w}) \vdash p(\overline{z})$, choose $C \subseteq M^*$ such that $\mathrm{tp}(AC, M^*) = q(\overline{z}, \overline{w})$.

Now write $\overline{x}_r$ as $\overline{yzw}$ and let $r(\overline{x}_r) = \mathrm{tp}(BAC, N^*)$. It is evident that the Striation Conditions are satisfied. As for $\overline{g}_r$, we can put $\overline{g}_r(t) = \overline{g}_q(t)$ for every $t \in u_q \cap P$. So, it only remains to define $g_{0,r}(t^*)$ and $g_{1,r}(t^*)$. The latter is easy, as we used N^* as our template. That is, define $g_{1,r}(t^*)$ to be the function mapping $\overline{yzw}$ to BAC.

Finally, the definition of $g_{0,r}(t^*)$ is where we use our assumption that u_q does not obstruct u_p at J. We are assuming that $t^* = \min(u_p \setminus J)$ and $P(t^*)$ holds. Choose $s \in u_p$ to be from the largest E-class in u_p below t^*. As $p \in \mathbb{Q}_I$, $\neg P(s)$ holds. Let $s^* = \max(u_q)$. As u_q does not obstruct u_p at J, $s^*/E \leq s/E$. Thus, there is no reason to extend $g_{0,p}(t^*)$, and we simply let $g_{0,r}(t^*) = g_{0,p}(t^*)$. $\square_{5.3.16}$

Lemma 5.3.17. *Suppose $p, q \in \mathbb{Q}_I$, $J \subseteq I$ is endless, $u_q \subseteq J$, $p\restriction_J \leq q$, and u_q does not obstruct u_p at J. Then there is $r \in \mathbb{Q}_I$ with $\overline{x}_r = \overline{x}_p \cup \overline{x}_q$, $r \geq p$, and $r \geq q$. Moreover, if q is a simple extension of $p\restriction_J$, then r is a simple extension of p.*

Proof. Arguing by induction on $|u_p \setminus J|$, this follows immediately from Lemma 5.3.16. $\square_{5.3.17}$

We now consider the 'harder half' where we do need to extend g_0.

Lemma 5.3.18. *Suppose $p, q \in \mathbb{Q}_I$, $t^* = \max(u_p)$, $P(t^*)$ holds, $u_q \subseteq I\restriction_{<t^*}$, and $p\restriction_{<t^*} \leq q$. Then there is $r \in \mathbb{Q}_I$, $r \geq p$, $r \geq q$, and $\max(u_r) = t^*$.*

Proof. Let $w^* = \max(u_q)$. If $w^*/E \leq s/E$ for some $s \in u_p$, $s < t^*$, then u_q would not obstruct u_p at $I\restriction_{<t^*}$ and we would be done by Lemma 5.3.16. So assume that $w^*/E > s/E$ for every $s \in u_p$, $s < t^*$. First, by Lemma 5.3.11, we may assume that q is s-topped. Arguing as in Case 2 of Lemma 5.3.16 write $\overline{x}_p$ as $\overline{yz}$, where $\overline{y}$ is $\overline{x}_p\restriction_{=t^*}$ and $\overline{z}$ is $\overline{x}_p\restriction_{<t^*}$. As well, write $\overline{x}_q$ as $\overline{z}, \overline{w}$ where $q(\overline{z}, \overline{w}) \vdash p(\overline{z})$. Again, $B = g_{1,p}(t^*)[\overline{y}] \subseteq N^* \setminus M^*$ and $A = g_{1,p}(t^*)[\overline{z}] \subseteq M^*$, where $\mathrm{tp}(BA, N^*) = p(\overline{y}, \overline{z})$. As $q(\overline{z}, \overline{w}) \vdash p(\overline{z})$, choose $C \subseteq M^*$

such that $\mathrm{tp}(AC, M^*) = q(\overline{z}, \overline{w})$. As AC is finite, choose $m \in \omega$ such that $AC \subseteq M_m$. Now consider $\mathrm{tp}(\overline{d}_m/AC, M_m)$. By applying Lemma 5.3.13, there is a simple extension q' of q such that $\mathrm{tp}(\overline{d}_m AC, M_m) = q'(\overline{x}_{q'})$. As q' is a simple extension of q, $\max(u_{q'}) < t^*$. Thus, by replacing q' by q, we may additionally assume that $\overline{d}_m \subseteq AC$. Finally, choose $s^* \in I$ satisfying $w^*/E < s^*/E < t^*$ and $\neg P(s^*)$.

We are now able to define r. Put $\overline{x}_r = \{x_{s^*,0}\} \cup \overline{yzw}$ and let $r(\overline{x}_r) = \mathrm{tp}(c_m BAC, N^*)$. For t^*, first let $g_{1,r}(t^*)$ map $\overline{x}_r$ onto $c_m BAC$. As we assumed $\overline{d}_m \subseteq AC$, $\overline{d}_m \subseteq range[g_{1,r}(t^*)]$. Finally, let $\ell_{t^*,r} = \ell + 1$, where $\ell = \ell_{t^*,p}$ and let $g_{0,r}(t^*)$ be the one-element end extension of $g_{0,p}(t^*)$ formed by $g_{0,r}(t^*)(\ell) = s^*$. It is easily verified that $r \in \mathbb{Q}_I$ is as desired. $\square_{5.3.18}$

Proposition 5.3.19. *Suppose $p, q \in \mathbb{Q}_I$, $J \subseteq I$ is endless, $u_q \subseteq J$, and $p\restriction_J \leq q$. Then there is $r \in \mathbb{Q}_I$ with $\max(u_r) = \max(u_p)$, $r \geq p$, and $r \geq q$.*

Proof. This follows immediately by induction on the number of E-classes of elements in $u_p \setminus J$, using either Lemma 5.3.16 or Lemma 5.3.18 at each step. $\square_{5.3.19}$

From this, we can easily verify that $\mathbb{Q}_I$ has the c.c.c.

Lemma 5.3.20. $(\mathbb{Q}_I, \leq_{\mathbb{Q}})$ *has the c.c.c.*

Proof. Let $\{p_i : i < \aleph_1\} \subseteq \mathbb{Q}_I$ be a collection of conditions. We will find $i \neq j$ for which p_i and p_j are compatible. We successively reduce this set maintaining its uncountability. By the Δ-system lemma we may assume that there is a single u^* such that for all i, j, $u_{p_i} \cap u_{p_j} = u^*$. Further, by the pigeonhole principle we can assume that for each $t \in u^*$, $n_{p_i,t} = n_{p_j,t}$. We can use pigeon-hole again to guarantee that all the p_i and p_j agree on the finite set of shared variables. Furthermore, by pigeon-hole, we may assume $\overline{g}_{p_i}(t) = \overline{g}_{p_j}(t)$ for all $t \in u^* \cap P$. And finally, since I is $\aleph_1$-like we can choose an uncountable set X of conditions such that for $i < j$ and $p_i, p_j \in X$ all elements of u^* precede anything in any $u_{p_i} \setminus u^*$ or $u_{p_j} \setminus u^*$ and that all elements of $u_{p_i} \setminus u^*$ are less that all elements of $u_{p_j} \setminus u^*$.

Finally, choose any $i < j$ from X. Let $J = \{s \in I : s < \min(u_{p_j} \setminus u_{p_i})\}$. By Proposition 5.3.19 applied to p_i and p_j for this choice of J, we conclude that p_i and p_j are compatible. $\square_{5.3.20}$

In the remainder of Section 5.3 we list the crucial constraints, which are sets of conditions, and we prove each of them to be dense and open in $\mathbb{Q}_I$. While A-C are quite similar to [BLS16]; the later ones depend more on this context. Before stating the first constraint we prove a lemma needed to study it.

Lemma 5.3.21. *For every $p \in \mathbb{Q}_I$ and every $t \in I$ such that $P(t)$ holds and $w < t$ for every $w \in u_p$, there is $q \in \mathbb{Q}_I$, $q \geq p$, with $\max(u_q) = t$ and $u_q \cap P = (u_p \cap P) \cup \{t\}$.*

Proof. Given p and t as assumed, first note that by Lemma 5.3.11, we may assume p is s-topped. Choose any B from M^* realizing $p(\overline{x}_p)$. We construct q as follows: $\overline{x}_q = \{x_{t,0}\} \cup \overline{x}_p$, and put $q(\overline{x}_q) = \mathrm{tp}(a^*B, N^*)$, where a^* is the 'preferred element' in $N^* \setminus M^*$. For $t' \in u_p \cap P$, take $\overline{g}_q(t') = \overline{g}_p(t')$. Let $g_{1,q}(t)$ be the elementary map from $\overline{x}_q$ onto a^*B, and let $g_{0,q}(t) = s$ for any choice of $s \in u_p$ for which s/E is maximal in u_p. It is easily verified that q is as required. $\square_{5.3.21}$

A. Surjectivity Our first group of constraints ensure that for any generic $G \subseteq \mathbb{Q}_I$, for every $(t,n) \in I \times \omega$, there is $p \in G$ such that $x_{t,n} \in \overline{x}_p$. To enforce this, for any $(t,n) \in I \times \omega$, let

$$\mathcal{A}_{t,n} = \{p \in \mathbb{Q}_I : x_{t,n} \in \overline{x}_p\}$$

Claim 5.3.22. *For every $(t,n) \in I \times \omega$, $\mathcal{A}_{t,n}$ is dense and open.*

Proof. Each of these sets are trivially open. We first show that $\mathcal{A}_{t,0}$ is dense for every $t \in I$. To see this, first recall that $\mathcal{A}_{\min(I),0} = \mathbb{Q}_I$ by Striation constraint 1(a). Next, fix $t \neq \min(I)$ and choose $p \in \mathbb{Q}_I$ arbitrarily. Take the endless proper initial segment $J = I_{<t}$ and let $q = p\restriction_J$. Using either Lemma 5.3.11 or Lemma 5.3.21 (depending on whether or not $P(t)$ holds) we get $q' \geq q$ with $u_{q'} \subseteq J$ and $q' \in \mathcal{A}_{t,0}$. Now, using Proposition 5.3.19, we get $r \geq p$ and $r \geq q'$. As $r \in \mathcal{A}_{t,0}$, we have shown $\mathcal{A}_{t,0}$ to be dense.

Next, we prove by induction on n that if $\mathcal{A}_{t,n}$ is dense, then so is $\mathcal{A}_{t,n+1}$. But this is trivial. Fix t and choose $p \in \mathbb{Q}_I$ arbitrarily. By our inductive hypothesis, there is $q \geq p$ with $x_{t,n} \in \overline{x}_q$. If $x_{t,n+1} \in \overline{x}_q$, there is nothing to prove, so assume otherwise. Then, necessarily, $n_{q,t} = n+1$. Let r be the extension of q formed by $\overline{x}_r = \overline{x}_q \cup \{x_{t,n+1}\}$ and $r(\overline{x}_r)$ the complete type generated by $q(\overline{x}_q) \cup \{x_{t,n+1} = x_{t,n}\}$. We take $g_{0,r} = g_{0,q}$ and for every $t' \in u_q \cap P$ with $t \leq t'$, let $g_{1,r}(t')$ be the one-point extension of $g_{1,q}(t')$ satisfying $g_{1,q}(t')(x_{t,n+1}) = g_{1,q}(t')(x_{t,n})$. $\square_{5.3.22}$

Next we describe the Henkin constraints.

B. Henkin For every $t \in I \setminus \{\min(I)\}$, finite sequence $\overline{z}$ from $I_{<t} \times \omega$ (in the sense of Definition 5.2.3) and complete formula $\varphi(y, \overline{z})$,

$\mathcal{B}_{t,\varphi} = \{p \in \mathbb{Q}_I : \overline{z} \subseteq \overline{x}_p$ and **either** $\neg\exists y \varphi(y,\overline{z}) \in p$ **or** $\varphi(x_{s,m}, \overline{z}) \in p$ for some $s < t, m < \omega\}$

Claim 5.3.23. *For every $t \in I \setminus \{\min(I)\}$ and complete $\psi(y,\overline{z})$, $\mathcal{B}_{\psi,t}$ is dense and open.*

Proof. Fix t and φ. Clearly, $\mathcal{B}_{t,\psi}$ is open. To show density, choose any $p \in \mathbb{Q}_I$. By applying Lemma 5.3.11, we may assume p is non-trivial. By iterating Claim 5.3.22, we may also assume $\overline{z} \subseteq \overline{x}_p$. There are now two cases: First, if $p \vdash \neg\exists y \psi(y,\overline{z})$, then $p \in \mathcal{B}_{t,\psi}$, so assume otherwise, i.e., $p(\overline{x}_p) \cup \{\psi(y,\overline{z})\}$ is consistent. Choose any complete formula $\varphi(y, \overline{x}_p)$ extending ψ that includes p. It follows immediately from Lemma 5.3.8 that there is a simple extension q of p with $q \in \mathcal{B}_{t,\psi}$. $\square_{5.3.23}$

In order to ensure our generic model is 'full' we need a minor variant of the Henkin constraints.

C. Fullness Suppose $\overline{z}$ is a finite sequence (in the sense of Definition 5.2.3), $t \in I$, and a formula $\varphi(y, \overline{z})$ satisfies $\varphi(y, \overline{z}) \vdash \theta(\overline{z})$ for some complete formula $\theta(\overline{z})$, but $\varphi(y, \overline{z})$ is not pseudo-algebraic.

$$\mathcal{C}_{\varphi,t} = \{p \in \mathbb{Q}_I : \text{ there is } s > t, s \in u_p, \overline{z} \subseteq \overline{x}_p, p \vdash \varphi(x_{s,0}, \overline{z})\}$$

Claim 5.3.24. *Each is $\mathcal{C}_{\varphi,t}$ is dense and open.*

Proof. Fix $\varphi(y, \overline{z})$ and t, and choose any $p \in \mathbb{Q}_I$. Fix any $s^* > t$. By extending p as needed, by Claim 5.3.22 we may assume $s^* \in u_p$ and $\overline{z} \subseteq \overline{x}_p$. As $\varphi(y, \overline{z})$ is not pseudo-algebraic, there is a complete, non-pseudo-algebraic $\psi(y, \overline{x}_p)$ extending φ and including p. By Lemma 5.3.8 there is a simple extension q of p and $s \geq s^*$ (hence $s > t$) such that $q \vdash \varphi(x_{s,0}, \overline{z})$. $\square_{5.3.24}$

D. $\mathbf{g_0}$ cofinal We introduce two sets of conditions that guarantee that in any generic G, $g_0(t)$ will describe an ω-chain that is cofinal in $I_{<t}$.

Fix $t \in I$ such that $P(t)$ holds.

- For each $s < t$, $\mathcal{D}_{t,s} = \{p \in \mathbb{Q}_I : t \in u_p$ and for some n, $s/E < g_{0,p}(t)(n)/E < t\}$;
- For every $n \in \omega$, $\mathcal{E}_{t,n} = \{p \in \mathbb{Q}_I : t \in u_p$ and $n \in \mathrm{dom}(g_{0,p}(t))\}$.

Claim 5.3.25. *For every $t \in I$ satisfying $P(t)$, every $s < t$, and every $n \in \omega$, both $\mathcal{D}_{t,s}$ and $\mathcal{E}_{t,n}$ are dense and open.*

Proof. That each set is open is immediate. Fix t such that $P(t)$ holds. We first argue that $\mathcal{D}_{t,s}$ is dense for every $s < t$. To see this, fix $s < t$ and choose any $p \in \mathbb{Q}_I$. By Claim 5.3.22 we may assume $t \in u_p$. Choose an endless initial segment J such that $t = \max(u_p \cap J)$. Let $q = p{\restriction_J}$ and let $q_1 = p{\restriction_{<t}}$. Note that $\max(u_q) = t$. Let $s^* = \max(u_{q_1} \cup \{s\})$. From the definition of $\mathbf{I}^*$, choose $t^* \in I$ such that $s^* < t^* < t$ and $P(t^*)$ holds.

We first find $r \geq q$ with $\max(u_r) = t$ and $r \in \mathcal{D}_{t,s}$. By Lemma 5.3.21, choose $q_2 \geq q_1$ with $\max(u_{q_2}) = t^*$. Apply Proposition 5.3.19 to q_2 and q to obtain an upper bound r satisfying $\max(u_r) = t$. As $r \in \mathbb{Q}_I$ while s is not in the maximal E-class of $u_r \setminus \{t\}$, by Condition 3(c) there is n such that $s/E < g_{0,r}(t)(n)$. That is, $r \in \mathcal{D}_{t,s}$. Finally, apply Proposition 5.3.19 again to p and r to get an extension of p that is in $\mathcal{D}_{t,s}$.

Next, we argue by induction on n that each set $\mathcal{E}_{t,n}$ is dense. We begin with $n = 0$. Given t and any $p \in \mathbb{Q}_I$, by Claim 5.3.22 there is $q \geq p$ with $t \in u_q$. As $q \in \mathbb{Q}_I$, Condition 3(a) implies that $q \in \mathcal{D}_{t,0}$.

Finally, assume $\mathcal{E}_{t,n}$ is dense. Choose any $p \in \mathbb{Q}_I$. Using the density of $\mathcal{E}_{t,n}$, we may assume $n \in \mathrm{dom}(g_{0,p}(t))$. If $(n+1) \in \mathrm{dom}(g_{0,p})$ then there is nothing to prove, so assume otherwise. That is, $\mathrm{dom}(g_{0,p}(t)) = \{0, \ldots, n\}$. Let $s = g_{0,p}(t)(n)$. Choose s' satisfying $s/E < s'/E < t$. As $\mathcal{D}_{t,s'}$ is dense, choose $q \geq p$ with $q \in \mathcal{D}_{t,s'}$. As $s' \in u_q$, s/E is not in the maximal E-class of u_q below t, hence $g_{0,q}(t)$ properly extends $g_{0,p}(t)$, so $q \in \mathcal{E}_{t,n+1}$. $\square_{5.3.25}$

F. Adjusting the level Suppose $t \in I$ such that $P(t)$ holds, $\overline{z}$ is a finite sequence (in the sense of Definition 5.2.3) from $I_{\leq t} \times \omega$, $w > t$ and $n \in \omega$. Then $\mathcal{F}_{t,\overline{z},w,n}$ consists of all $p \in \mathbb{Q}_I$ such that $\{x_{w,n}\} \cup \overline{z} \subseteq \overline{x}_p$ and **either** p 'says' $x_{w,n} \notin \mathrm{pcl}(\overline{z})$ **or** p 'says' $x_{w,n} = x_{s,m}$ for some $s \leq t$, $m \in \omega$.

Claim 5.3.26. *For all $t, \overline{z}, w, n$ as above, $\mathcal{F}_{t,\overline{z},w,n}$ is dense and open.*

Proof. As always, 'open' is clear. To establish density, choose any $p \in \mathbb{Q}_I$. By iterating Claim 5.3.22 we may assume $\{x_{w,n}\} \cup \overline{z} \subseteq \overline{x}_p$. There are now two cases. If p 'says' $x_{w,n} \notin \mathrm{pcl}(\overline{z})$ we are done, so assume otherwise. Let $\varphi(y, \overline{x}_p)$ be the complete formula including p that extends '$y = x'_{w,n}$. Now apply Lemma 5.3.8 to get a simple extension q of p with $\overline{x}_q = \{x_{s,m}\} \cup \overline{x}_p$ and $\varphi(x_{s,m}, \overline{x}_p)$. Note that since $x_{w_n} \in \mathrm{pcl}(\overline{z})$, we have that $\varphi_t(y, \overline{x}_p\restriction_{\leq t})$ is pseudo-algebraic. Thus, $s \leq t$ as required to show $q \in \mathcal{F}_{t,\overline{z},w,n}$. $\square_{5.3.26}$

We now verify that the forcing $(\mathbb{Q}_I, \leq_\mathbb{Q})$ satisfies the conclusions of Proposition 5.2.2. Suppose $G \subseteq \mathbb{Q}_I$ is a filter meeting every dense open subset. Let

$$X[G] = \bigcup \{p(\overline{x}_p) : p \in G\}$$

Because of the dense subsets $\mathcal{A}_{t,n}$, $X[G]$ describes a complete type in the variables $\{x_{t,n} : t \in I, n \in \omega\}$. There is a natural equivalence relation $\sim_G$ on $X[G]$ defined by

$$x_{t,n} \sim_G x_{s,m} \qquad \text{if and only if} \qquad X[G] \text{ 'says' } x_{t,n} = x_{s,m}$$

Let $N[G]$ be the τ-structure with universe $X[G]/\sim_G$. As notation, for each t, n, let $a_{t,n} \in N[G]$ denote $[x_{t,n}] = x_{t,n}/\sim_G$. As each $p \in \mathbb{Q}_I$ describes a complete (principal) formula with respect to T, $N[G]$ is an atomic set. As well, it follows from Claim 5.3.23 that $N[G] \models T$.

For each $t \in I$ such that $P(t)$ holds, let $N_t = \{[x_{w,n}] : \text{some } x_{s,m} \in [x_{w,n}] \text{ with } s < t\}$. Similarly, for each $s \in I \setminus \{\min(I)\}$ with $\neg P(s)$, let $N_s = \{[x_{w,n}] : w/E < s/E\}$.

By repeated use of Claim 5.3.23, each N_t and N_s are elementary substructures of $N[G]$. Note that $N_{s'} = N_s$ whenever $E(s', s)$.

Given any (w, n), if there is a least $s \in I$ such that $a_{w,n} = a_{s,m}$ for some $m \in \omega$, then we say $a_{w,n}$ ***is on level*** s. For an arbitrary (w, n), a least s need not exist, but it does in many cases. Recall that the Striation constraints imply that for every $w \in I$, $a_{w,0}$ is on level w. As well, for any $n > 0$, $a_{t,n}$ is on level t for any t such that $P(t)$ holds. Because of the Level constraints (group **F**) for any t such that $P(t)$ holds, if $b \in N[G]$ and $b \in \mathrm{pcl}(\{a_{s,m} : s \le t, m \in \omega\}, N[G])$, $b = [x_{s',m'}]$ for some $m' \in \omega$ and $s' \le t$.

As $|I| = \aleph_1$ and the fact that each $a_{t,0} \notin \mathrm{pcl}(N_t, N[G])$, $||N[G]|| = \aleph_1$. Finally, it follows from the density of the 'Fullness conditions' that $N[G]$ is full.

More information about $N[G]$ can be gleaned from the functions g_1 and g_0. Fix any t such that $P(t)$ holds. Let $B_t \subseteq N[G]$ consist of all elements at level exactly t. Define

$$g_1^*(t) : B_t \cup N_t \to N^*$$

by $g_1^*(t)(a_{s,m}) = g_{1,p}(t)(x_{s,m})$ for some $p \in G$. As each map $g_{1,p}(t)$ is elementary and $g_{1,p}(t) \subseteq g_{1,q}(t)$ whenever $q \ge p$, this is well-defined. Because of Constraints 4(c,d), g_1^* maps N_t into M^* and takes B_t into $N^* \setminus M^*$. Furthermore, because of Lemma 5.3.12, g_1^* maps onto N^*. It follows that $B_t \cup N_t$ is the universe of an elementary submodel of $N[G]$ that is isomorphic to N^* via g_1^*. Similarly, the restriction of $g_1^*(t)$ to N_t is onto, hence yields an isomorphism between N_t and M^*. Finally, the restriction of $g_1^*(t)$ to B_t is onto $N^* \setminus M^*$. Also, by Constraint 4(b), $g_1^*(t)(a_{t,0}) = a^*$.

Similarly, because of density groups $\mathcal{D}$ and $\mathcal{E}$, the function

$$g_0^*(t) : \omega \to I_{<t}$$

defined by $g_0^*(t)(i) = g_{0,p}(t)(i)$ for some $p \in G$ is well-defined. Moreover, if we let s_i denote $g_0^*(t)(i)$, then the sequence $\langle s_i : i \in \omega \rangle$ is cofinal in $I_{<t}$ and satisfies $s_i/E < s_{i+1}/E$ for all i. By Constraint group 5 and our comments about $g_1^*(t)$ above, for every $i > 0$ there is $m(i) \in \omega$, $\bar{b}$ from N_{s_i} such that

- $g_1^*(t)(\bar{b}) = \bar{d}_m \subseteq M^*$; and
- $g_1^*(t)(a_{s_i,0}) = c_m$

As $s_i/E < s_{i+1}/E$, $a_{s_i,0} \in N_{s_{i+1}}$. As well, as the Striation constraints imply that $a_{s_i,0}$ is at level s_i, $a_{s_i,0} \notin N_{s_i}$. Finally, using $g_1(t)^{-1}$ the relation

$$c_m \in \mathrm{pcl}(\bar{d}_m a^*, N^*)$$

translates into

$$a_{s_i,0} \in \mathrm{pcl}(\bar{b} a_{t,0}, N[G])$$

It remains to verify that $N[G]$ satisfies the three conditions of Proposition 5.2.2. (1) is handled by the Henkin constraints, most notably Claim 5.3.23.

Towards (2), the argument just given implies that $a_{t,0}$ admits a cofinal chain in N_t for every t such that $P(t)$ holds. To complete the verification of (2) we show that $a_{t,0}$ also rk ∞-catches N_t whenever $P(t)$ holds. Fix such a t. We know that $N_t \preceq N[G]$, so it is pseudo-algebraically closed. Choose any $b \in N[G]$ such that $b \in \mathrm{pcl}(a_{t,0}N_t, N[G]) \setminus N_t$. Because of the Level constraints (group $\mathcal{F}$) we have that $b = a_{w,n}$ for some $w \leq t$. However, if $w < t$, then we would have $b \in N_t$, which it isn't. Thus, b is of level t, hence $b \in B_t$ in the notation defined above. Applying $g_1^*(t)$ to $a_{t,0}, N_t, b$ yields:

$$e \in \mathrm{pcl}(a^* M^*, N^*) \setminus M^*$$

where $e = g_1^*(t)(b)$. By Data 5.0.1(2) of the initial data, this implies $\mathrm{rk}(a^*/M^*e) < \infty$. Translating back via $g_1^*(t)$ yields $\mathrm{rk}(a_{t,0}/N_t b) < \infty$. Thus, $a_{t,0}$ rk ∞-catches N_t.

It remains to verify (3) of Proposition 5.2.2. Choose a seamless $J \subseteq I$ and let $N_J := \bigcup\{N_t : t \in J\} \preceq N[G]$. Choose any $b \in N[G] \setminus N_J$ that rk ∞ catches N_J, and we show that b has bounded effect in N_J. Say b is $[x_{w^*,n}]$, where necessarily $w^* \in I \setminus J$. By the fundamental theorem of forcing, there is $p \in G$ such that

$$p \Vdash [x_{w^*,n}]_{\tilde{\mathbf{G}}} \text{ rk } \infty\text{-catches } N_J[\tilde{\mathbf{G}}]. \qquad (*)$$

Thus, among other things, $p \Vdash$ '$x_{w^*,n} \neq x_{s,m}$' for all $s \in J$, $m \in \omega$. Choose any $s^* \in J$ such that $s^* > s$ for every $s \in u_p \cap J$. (Recall u_p from just below Definition 5.2.3.) That b has bounded effect in N_J follows immediately from the following Claim.

Claim 5.3.27. $p \Vdash \mathrm{pcl}(\{[x_{w^*,n}]\} \cup N_{<s^*}[\tilde{\mathbf{G}}], N[\tilde{\mathbf{G}}]) \cap N_J[\tilde{\mathbf{G}}] \subseteq N_{<s^*}[\tilde{\mathbf{G}}]$.

Proof. If not, then there is $q \in \mathbb{Q}_I$ satisfying $q \geq p$ and a finite $A = \{x_{w_i,m_i} : i < k\} \subseteq I_{<s^*} \times \omega$ such that, letting $A_{\tilde{\mathbf{G}}} = \{[x_{w_i,m_i}]_{\tilde{\mathbf{G}}} : i < k\}$,

$$q \Vdash \mathrm{pcl}(A_{\tilde{\mathbf{G}}}[x_{w^*,n}]_{\tilde{\mathbf{G}}}, N[\tilde{\mathbf{G}}]) \cap N_J[\tilde{\mathbf{G}}] \not\subseteq N_{<s^*}[\tilde{\mathbf{G}}]$$

Without loss, we may assume that for each $x_{w_i,m_i} \in A$, then $w_i \in u_q$. As J is seamless, by Lemma 5.1.3, choose an automorphism π of $(I, <, E, P)$ such that $\pi\restriction_{\geq \min(u_p \setminus J)} = id$; $\pi(t^*) = t^*$; $\pi\restriction_{u_p} = id$; $\pi\restriction_{u_q \cap I_{<s^*}} = id$, but $\pi(s^*) \notin J$. By Lemma 5.3.2, π extends to an automorphism π' of $\mathbb{Q}_I$ given by $x_{t,m} \mapsto x_{\pi(t),m}$. By our choice of π, $\pi'(p) = p$. Whereas $\pi'(q)$ need not equal q, we do have $p \leq \pi'(q)$.

$$\pi'(q) \Vdash \mathrm{pcl}(A_{\tilde{\mathbf{G}}}[x_{w^*,n}]_{\tilde{\mathbf{G}}}, N[\tilde{\mathbf{G}}]) \cap N_{\pi(J)}[\tilde{\mathbf{G}}] \not\subseteq N_{<\pi(s^*)}[\tilde{\mathbf{G}}] \qquad (**)$$

We argue that this forcing statement contradicts the just defined $(*)$. To see this, choose a generic $H \subseteq \mathbb{Q}_I$ with $\pi'(q) \in H$. As $p \leq \pi'(q)$, we also have $p \in H$. As above, let

$N[H] \in \mathbf{At_T}$ be the structure with universe $X/\sim_H$. Put $N_J[H] := \bigcup\{N_t[H] : t \in J\} \preceq N[H]$ and $N_{\pi(J)}[H] := \bigcup\{N_t[H] : t \in \pi(J)\} \preceq N[H]$. As notation, let $b_H := [x_{w^*,n}]_H$ and $A_H := \{[x_{w_i,m_i}]_H : i < k\}$. Now:

- $A_H \subseteq N_J[H]$.
- Applying π' to the statement $(*)$, along with $\pi'(p) = p$ and $\pi(w^*) = w^*$ yields

$$p \Vdash [x_{w^*,n}]_{\tilde{\mathbf{G}}} \text{ rk } \infty\text{-catches } N_{\pi(J)}[\tilde{\mathbf{G}}].$$

  As $p \in H$ and recording only half of the definition of 'rk ∞-catches' yields

$$\mathrm{rk}(b_H/N_{\pi(J)}, N[H]) = \infty$$

- From $(**)$, choose $e \in N[H]$ such that $e \in \mathrm{pcl}(A_H b_H, N[H])$ and $e \in N_{\pi(J)}[H]$, but $e \notin N_{<\pi(s^*)}[H]$. Thus, since $J \subseteq I_{<\pi(s^*)}$, $e \notin N_J[H]$.

However, since $\{e\} \cup N_J[H] \subseteq N_{\pi(J)}[H]$, we conclude $\mathrm{rk}(b_H/N_J[H]e, N[H]) = \infty$. Combining this with $e \in \mathrm{pcl}(N_J[H]b_H, N[H])$ and $e \notin N_J[H]$, we contradict 'b_H rk ∞-catches $N_J[H]$.' $\square_{5.3.27}$

Finally, as Claim 5.3.27 holds for any sufficiently large $s^* \in J$, $b = a_{w^*,n}$ has bounded effect in N_J. This establishes (3), and thus concludes the proof of Proposition 5.2.2. $\square_{5.2.2}$

5.4 Many non-isomorphic models in $\mathbf{At_T}$

We continue to work under the assumption of Data 5.0.1 and the notation there. With Proposition 5.4.1 below, we prove the existence of $2^{\aleph_1}$ non-isomorphic atomic models, each of size $\aleph_1$, under the assumption that a countable, transitive model (M, ϵ) of ZFC exists. The main theorem of this section, Theorem 5.4.2, follows easily from this.

Proposition 5.4.1. *Assume that a countable, transitive model of ZFC exists. If we have an instance of Data 5.0.1, then there are atomic models ($N_X : X \subseteq \omega_1$) such that $N_X \ncong N_Y$ whenever $X \triangle Y$ is stationary.*

Proof. Choose $N^*, M^*, a^*, \bar{c}_m, c_m$ witnessing Data 5.0.1 and fix a countable, transitive model (M, ϵ) of ZFC containing these sets, along with T and τ. We begin by working inside M. In particular, choose $S \subseteq \omega_1^M$ such that

$$(M, \epsilon) \models \text{'}S \text{ is stationary/costationary'}$$

Now perform Construction 5.1.4 inside M to obtain $I = (I^S, <, P, E) \in \mathbf{I}^*$.

Next, we force with the c.c.c. poset $\mathbb{Q}_{I^S}$ and find $(M[G], \epsilon)$, where G is a generic subset of $\mathbb{Q}_{I^S}$. As the forcing is c.c.c., it follows that all cardinals are preserved, as well as stationarity. Thus, $\omega_1^{M[G]} = \omega_1^M$ and $(M[G], \epsilon) \models$ 'S is stationary/costationary'. As well, $(I^S)^{M[G]} = I^S$. According to Proposition 5.2.2, inside $M[G]$ there is an atomic, full $N_I \models T$ that is striated according to $(I^S, <, P, E)$. Write the universe of N_I as $\{a_{t,n} : t \in I^S, n \in \omega\}$. Inside $M[G]$ we have the mapping $\alpha \mapsto J_\alpha$ given by Construction 5.1.4. For every $\alpha \in \omega_1^{M[G]}$, let N_α be the τ-substructure of N_I with universe $\{a_{t,n} : t \in J_\alpha, n \in \omega\}$. It follows from Proposition 5.2.2 and Construction 5.1.4 that for every non-zero $\alpha \in \omega_1^{M[G]}$,

- $N_\alpha \preceq N_I$;
- If $\alpha \in S$, then $I^S \setminus J_\alpha$ has a least E-class t/E which is topped with $t^* = top(t/E)$, and $a_{t^*,0}$ both rk ∞-catches and admits a cofinal chain in N_α; and
- If $\alpha \notin S$, then every $b \in N_I \setminus N_\alpha$ that rk ∞-catches N_α has bounded effect in N_α.

Now, still working inside $M[G]$, we form a 3-sorted structure N^* that encodes this information. The language of N^*[1] will be

$$\tau^* = \tau \cup \{U, V, W, <_U, <_V, P, E, R_1, R_2, F\}$$

N^* is the τ^*-structure in which

- $\{U, V, W\}$ are unary predicates that partition the universe;
- $(U^{N^*}, <_U)$ is $(\omega_1^{M[G]}, <)$;
- $(V^{N^*}, <_V, P, E)$ is $(I^S, <, P, E)$;
- W^{N^*} is N_I (the functions and relations in τ only act on the W-sort);
- $R_1 \subseteq U \times V$, with $R_1(\alpha, t)$ holding if and only if $t \in J_\alpha$;
- $R_2 \subseteq U \times W$, with $R_2(\alpha, b)$ holding if and only if $b \in N_\alpha$; and
- $F : W \times U \times \omega \to V$ satisfies: For every b and for every limit ordinal α, $\langle F(b, \alpha, n) : n \in \omega \rangle$ is a strictly increasing, cofinal sequence in J_α.

[1] Note the inclusion of F!

Note that because of E, $S \subseteq \omega_1^{M[G]}$ is an τ^*-definable subset of the U-sort of N^*. Also, on the W-sort, the relation '$b \in \mathrm{pcl}(\overline{a})$' is definable by an infinitary τ^*-formula. Thus, the relations 'b rk ∞-catches N_α', 'b has bounded effect in N_α', and '$\langle F(b, \alpha, n) : n \in \omega \rangle$ witnesses that b admits a cofinal chain' are all infinitarily τ^*-definable subsets of $U \times W$.

By construction, $N^* \models \psi$, where the infinitary ψ asserts:

> For every non-zero $\alpha \in U$, **either** there is an element of W^{N^*} that rk ∞-catches N_α and admits a cofinal chain in N_α **or** every element of W^{N^*} that rk ∞-catches N_α has bounded effect in N_α.

Fix an infinitary τ^*-formula $\theta(x)$ such that for x from the U-sort, $\theta(x)$ holds if and only if there exists $b \in N_I \setminus N_{J_x}$ that rk ∞-catches and has a cofinal chain in N_{J_x}. Thus, for $\alpha \in \omega_1^{M[G]}$ we have

$$N^* \models \theta(\alpha) \quad \Longleftrightarrow \quad \alpha \in S$$

Fix a countable fragment L_A of $L_{\omega_1,\omega}(\tau^*)$ to include the formulas mentioned above, along with infinitary formulas ensuring τ-atomicity.

Now, we switch our attention to V. By applying Theorem 3.13 of [BLS16], which is proved by the method of iterated M-normal ultrapowers, to $(M[G], \epsilon)$, L_A, and N^*, we obtain (in V!) a family (M_X, E) of elementary extensions of $(M[G], \epsilon)$, each of size $\aleph_1$, indexed by subsets $X \subseteq \omega_1$ ($= \omega_1^V$). Each of these models of ZFC has an L^*-structure, which we call N_X^* inside it. As well, for each $X \subseteq \omega_1$, there is a continuous, strictly increasing mapping $t_X : \omega_1 \to U^{N_X^*}$ with the property that

$$N_X^* \models \theta(t_X(\alpha)) \quad \Longleftrightarrow \quad \alpha \in X$$

Let $(I^X, <, E, P)$ be the 'V-sort' of N_X^*. Clearly, each $I^X \in \mathbf{I}^*$.

Finally, the W-sort of each τ^*-structure N_X^* is an τ-structure, striated by I^X. We call this 'reduct' N_X. Note that by our choice of L_A and the fact that $N_X^* \succeq_{L_A} N^*$, we know that every τ-structure N_X is an atomic model of T, which is easily seen to be of cardinality $\aleph_1$.

Thus, it suffices to prove that there is no τ-isomorphism $f : N_X \to N_Y$ whenever $X \triangle Y$ is stationary. For this, choose $X, Y \subseteq \omega_1$ such that $X \setminus Y$ is stationary and by way of contradiction assume that $f : N_X \to N_Y$ were an τ-isomorphism. Each of N_X, N_Y has its 'expansion' to L^*-structures N_X^* and N_Y^*, respectively. As notation, for each $\alpha \in \omega_1^V$, let N_α^X and N_α^Y denote τ-elementary substructures with universes of $R_2(t_X(\alpha), N_X^*)$ and $R_2(t_Y(\alpha), N_Y^*)$, respectively.

Next, choose a club $C_0 \subseteq \omega_1$ such that for every $\alpha \in C_0$:

- α is a limit ordinal;

- The restriction of $f : N_\alpha^X \to N_\alpha^Y$ is a τ-isomorphism.

Put $C := \lim(C_0)$. As C is club and $(X \backslash Y)$ is stationary, choose α in their intersection. Fix a strictly increasing ω-sequence $\langle \alpha_n : n \in \omega \rangle$ of elements from C_0 converging to α. As $\alpha \in X$, we can choose an element $b \in N_X \setminus N_\alpha^X$ and a strictly increasing sequence $\langle s_m : m \in \omega \rangle$ converging to α such that b rk ∞-catches N_α^X and for every $m \in \omega$

$$\mathrm{pcl}(N_{s_m}^X \cup \{b\}) \cap N_{s_{m+1}}^X \not\subseteq N_{s_m}^X.$$

As the sets J_{α_n} are all proper initial segments of J_α with $\bigcup J_{\alpha_n} = J_\alpha$, there is an integer k such that for all $n \geq k$, there is an integer $m(n)$ such that $s_m \in J_{\alpha_n}$, but $s_{m+1} \notin J_{\alpha_n}$. Thus, for any $n \geq k$,

$$\mathrm{pcl}(N_{\alpha_n}^X \cup \{b\}) \cap N_\alpha^X \not\subseteq N_{\alpha_n}^X.$$

But now, as '$b \in \mathrm{pcl}(\overline{a})$' is preserved under τ-isomorphisms and $f[N_{\alpha_n}^X] = N_{\alpha_n}^Y$ setwise, we have that $f(b)$ rk ∞-catches N_α^Y, but for every $n \geq k$,

$$\mathrm{pcl}(N_{\alpha_n}^Y \cup \{f(b)\}) \cap N_\alpha^Y \not\subseteq N_{\alpha_n}^Y.$$

From this, as pcl is finitely based, it follows easily that for every $s^* \in J_\alpha$, there is $s \in J_\alpha$, $s > s^*$ such that

$$\mathrm{pcl}(N_s^Y \cup \{f(b)\}, N_Y) \cap N_\alpha^Y \neq N_s^Y$$

That is, $f(b)$ does not have bounded effect in N_α^Y. As $\alpha \notin Y$, we obtain a contradiction from $N_Y^* \models \neg\theta(t_Y(\alpha))$ and $N_Y^* \models \psi$. □

Theorem 5.4.2. *If* $\mathbf{At_T}$ *has* $< 2^{\aleph_1}$ *non-isomorphic atomic models of size* $\aleph_1$*, then* $\mathbf{At_T}$ *is ranked.*

Proof. Assume that $\mathbf{At_T}$ is not ranked. By Proposition 3.16 we obtain a witness to Data 5.0.1. As the proof of Proposition 5.4.1 is finite, the hypotheses there can be weakened to the existence of a countable, transitive model of a large enough, finite subset of ZFC. However, the existence of such a countable, transitive model is provable from ZFC itself (using the Reflection Theorem). Thus, the existence of $2^{\aleph_1}$ non-isomorphic atomic models of size $\aleph_1$ is provable in ZFC alone. □

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