A trichotomy of countable, stable, unsuperstable theories

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Abstract

Every countable, strictly stable theory either has the Dimensional Order Property (DOP), is deep, or admits an 'abelian group witness to unsuperstability'. To obtain this and other results, we develop the notion of a 'regular ideal' of formulas and study types that are minimal with respect to such an ideal.

1 Introduction

By definition, a stable, unsuperstable theory (henceforth called *strictly stable*) admits a type that is not based on any finite subset of its domain. From this

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one sees that such a theory admits trees of definable sets. That is, there is a sequence $\langle \varphi_n(x,y) : n \in \omega \rangle$ of formulas such that for any cardinal κ there are definable sets $\{\varphi_n(x,a_\nu) : \nu \in {}^{<\omega}\kappa\}$ giving rise to $\kappa^{\aleph_0}$ partial types $\{p_\mu : \mu \in {}^{\omega}\kappa\}$ where each p_μ forks over $\{a_{\mu|k} : k < n\}$ for all $n \in \omega$. In [12] the second author used these trees to count the number of uncountable models or to find the maximal size of a family of pairwise nonembeddable models of a fixed cardinality of any stable, unsuperstable theory. However, for other combinatorial questions, such as computing the Karp complexity of the class of uncountable models of such a theory, the existence of these trees does not seem to be sufficient. Here, with Theorem 4.3, we prove that when the language is countable, any strictly stable theory exhibits at least one of the three more detailed nonstructural properties mentioned in the abstract. This theorem is used in [9], but it is likely to be applicable to other contexts as well. Two of the alternatives, the Dimensional Order Property (DOP) or a theory being deep appear in [12] and are compatible with superstability. The third alternative is new and is captured by the following definition:

Definition 1.1 An *abelian group witness to unsuperstability* is a descending sequence $\langle A_n : n \in \omega \rangle$ of abelian groups with $[A_n : A_{n+1}]$ infinite for each n such that the intersection $A = \bigcap_n A_n$ is connected and whose generic type is regular.

The existence of such a sequence readily contradicts superstability as for any cardinal κ, one immediately obtains a family of $\{C_\mu : \mu \in {}^{\omega}\kappa\}$ of cosets of A indexed by a tree. The family of generic types of these cosets form a counterexample to superstability witnessed by regular types. As well, with Theorems 4.3 and 4.4 we see that one can frequently say more about the generic type of A. This added information is used in [9].

As examples, the theory of separably closed fields has DOP ([1] or [2]). The theory of infinitely many, refining equivalence relations in which each E_{n+1} splits each E_n-class into infinitely many classes is strictly stable and deep, as is the model completion of a single unary function. For the third case, the theory of $(\mathbb{Z}^\omega, +, U_n)_{n\in\omega}$, where each U_n is the subgroup in which the first n coordinates are zero, is the paradigm of an abelian group witness to unsuperstability.

In order to establish these theorems, much of the paper discusses the notion of a *regular ideal* of formulas (see Definition 2.3). The origins of these ideas date back to Section V.4 of [12] and have been reworked and expanded in [3] and [10]. As well, the proof of the trichotomy depends on results from [8].

Our notation is standard, and complies with either [10] or [12]. For a stable theory T $\kappa_r(T)$ denotes the least regular cardinal κ such that there is no forking chain of length κ. Thus, a stable theory is superstable if and only if $\kappa_r(T) = \aleph_0$ and $\kappa_r(T) = \aleph_1$ when T is countable and strictly stable. We call a model 'a-saturated' (a-prime) in place of '$\mathbf{F}^a_{\kappa_r(T)}$-saturated' ($\mathbf{F}^a_{\kappa_r(T)}$-prime).

Throughout the whole of this paper we assume '$\mathbf{T} = \mathbf{T}^{\mathrm{eq}}$.' That is, T is a stable theory in a multi-sorted language, $\mathfrak{C}$ is a large, saturated model of T, and the language L is closed under the following operation: If $E(\bar{x}, \bar{y})$ is a definable equivalence relation then there is a sort U_E and a definable surjection $f_E : \mathfrak{C}^{lg(\bar{x})} \to U_E(\mathfrak{C})$ in the language L. In particular, the set of sorts is closed under finite products. Thus any finite tuple of elements from varying sorts can be viewed as an element of the product sort. With this identification, every formula can be considered to have a single free variable. As notation, $L(\mathfrak{C})$ denotes the set of formulas with parameters from $\mathfrak{C}$ and for a specific sort s, $L_s(\mathfrak{C})$ denotes the $L(\mathfrak{C})$-formulas $\varphi(x)$ in which the free variable has sort s.

2 Regular ideals

Definition 2.1 An *invariant ideal* $\mathcal{ID}$ is a subset of $L(\mathfrak{C})$ containing all algebraic formulas, closed under automorphisms of $\mathfrak{C}$, and for any sort s and any $\varphi, \psi \in L_s(\mathfrak{C})$

1. If $\varphi, \psi \in \mathcal{ID}$ then $\varphi \vee \psi \in \mathcal{ID}$; and
2. If $\varphi \vdash \psi$ and $\psi \in \mathcal{ID}$, then $\varphi \in \mathcal{ID}$.

A partial type Γ (i.e., a subset of $L_s(\mathfrak{C})$ for some sort s) is $\mathcal{ID}$-*small* if it entails some element of $\mathcal{ID} \cap L_s(\mathfrak{C})$.

Examples of invariant ideals include the algebraic formulas, the superstable formulas (see Definition 4.1) or the ideal of formulas with Morley rank. These and other examples are discussed in [10]. Many times we will make use of the fact that formulas in $\mathcal{ID}$ may have 'hidden' parameters.

Lemma 2.2 *Let $\mathcal{ID}$ be any invariant ideal.*

1. *A complete type $p \in S(A)$ is $\mathcal{ID}$-small if and only if $p \cap \mathcal{ID} \neq \emptyset$.*

2. *For any A and a,* $\mathrm{stp}(a/A)$ *is $\mathcal{ID}$-small if and only if* $\mathrm{tp}(a/A)$ *is $\mathcal{ID}$-small.*

3. *If $A \subseteq B$ and* $\mathrm{tp}(a/B)$ *does not fork over A, then* $\mathrm{tp}(a/A)$ *is $\mathcal{ID}$-small if and only if* $\mathrm{tp}(a/B)$ *is $\mathcal{ID}$-small.*

Proof. (1) Right to left is immediate. For the converse, assume p entails $\psi \in \mathcal{ID}$. By compactness there is $\varphi \in p$ such that $\varphi \vdash \psi$, hence $\varphi \in \mathcal{ID}$.

(2) Right to left is clear. If $\mathrm{stp}(a/A)$ entails $\psi(x, b) \in \mathcal{ID}$, then by compactness and the finite equivalence relation theorem there is an A-definable equivalence relation $E(x, y)$ with finitely many classes such that $\mathrm{tp}(a/A) \cup \{E(x, c)\} \vdash \psi(x, b)$ for some c. Choose A-automorphisms $\{\sigma_i : i < n\}$ of $\mathfrak{C}$ such that $\{E(x, \sigma_i(c)) : i < n\}$ includes all the E-classes. Since $\mathcal{ID}$ is an invariant ideal $\bigvee_{i<n} \psi(x, \sigma_i(b)) \in \mathcal{ID}$ and $\mathrm{tp}(a/A) \vdash \bigvee_{i<n} \psi(x, \sigma_i(b))$.

(3) By (2) it suffices to prove this for strong types. Assume $\mathrm{stp}(a/B)$ is $\mathcal{ID}$-small. By (1) and (2), choose $\psi(x, b) \in \mathrm{tp}(a/B) \cap \mathcal{ID}$. Choose $\{b_i : i \in \kappa(T)\}$ independent over A, each having the same strong type over A as b. Since $\mathcal{ID}$ is invariant, $\psi(x, b_i) \in \mathcal{ID}$ for each i. Furthermore, since any a' realizing $\mathrm{stp}(a/A)$ is independent with some b_i over A, ab and $a'b_i$ realize the same strong type over A, hence $\psi(a', b_i)$ holds. By compactness, there is a finite subset F such that $\mathrm{stp}(a/A) \vdash \bigvee_{i \in F} \psi(x, b_i)$, so $\mathrm{stp}(a/A)$ is $\mathcal{ID}$-small.

Definition 2.3 An invariant ideal $\mathcal{ID}$ is *regular* if, for all $L(\mathfrak{C})$-formulas $\psi(y)$ and $\theta(x, y)$, if $\psi \in \mathcal{ID}$ and $\theta(x, b) \in \mathcal{ID}$ for every $b \in \psi(\mathfrak{C})$, then the formula $\exists y(\psi(y) \wedge \theta(x, y)) \in \mathcal{ID}$.

By analogy, recall that a cardinal κ is regular if and only if the union of fewer than κ sets, each of size less than κ, has size less than κ.

We call a strong type $\mathrm{stp}(a/A)$ *$\mathcal{ID}$-internal* if there is a set $B \supseteq A$ independent from a over A, a B-definable function f, and elements $\bar{c}$ such that $\mathrm{tp}(c/B)$ is $\mathcal{ID}$-small for each $c \in \bar{c}$ and $a = f(\bar{c})$. The strong type $\mathrm{stp}(a/A)$ is *$\mathcal{ID}$-analyzable* if there is a finite sequence $\langle a_i : i \leq n \rangle$ from $\mathrm{dcl}(Aa)$ such that $a_n = a$ and $\mathrm{stp}(a_i/A \cup \{a_j : j < i\})$ is $\mathcal{ID}$-internal for each

$i \leq n$. Since $\mathcal{ID}$ is a collection of formulas, this definition of analyzability is equivalent to the usual one, see e.g., [10].

In order to iterate the defining property of a regular ideal, we need the following notion, whose terminology is borrowed from [6].

Definition 2.4 A formula $\varphi(x, c)$ is *in $\mathcal{ID}$, provably over B* if there is some $\theta(y) \in \mathrm{tp}(c/B)$ such that $\varphi(x, c') \in \mathcal{ID}$ for every c' realizing θ.

Lemma 2.5 *For all sets B and every $n \in \omega$, if $\varphi(x, y_0, \ldots, y_{n-1})$ is B-definable and $a, c_0, \ldots, c_{n-1}$ satisfy:*

1. $\mathrm{tp}(c_i/B)$ *is $\mathcal{ID}$-small for each $i < n$;*
2. $\varphi(x, c_0, \ldots, c_{n-1}) \in \mathcal{ID}$ *provably over B; and*
3. $\varphi(a, c_0, \ldots, c_{n-1})$

then $\mathrm{tp}(a/B)$ *is $\mathcal{ID}$-small.*

Proof. Fix any set B. We argue by induction on n. If $n = 0$ the formula $\varphi(x)$ itself witnesses that $\mathrm{tp}(a/B)$ is $\mathcal{ID}$-small. Assume the result holds for n and fix a formula $\varphi(x, c_0, \ldots, c_n)$ and $a, c_0, \ldots, c_n$ satisfying Conditions (1)–(3). By (1) and Lemma 2.2, choose $\psi(y_n) \in \mathrm{tp}(c_n/B) \cap \mathcal{ID}$. By (2), choose $\theta \in \mathrm{tp}(c_0, \ldots, c_n/B)$ such that $\varphi(x, c'_0, \ldots, c'_n) \in \mathcal{ID}$ for all $c'_0, \ldots, c'_n$ realizing θ. By shrinking θ we may assume that $\theta(y_0, \ldots, y_n) \vdash \psi(y_n)$.

Let $\theta^* := \exists y_n \theta$ and $\varphi^* := \exists y_n(\varphi \wedge \theta)$. By our inductive hypothesis, in order to prove that $\mathrm{tp}(a/B)$ is $\mathcal{ID}$-small, it suffices to show that φ^* and $a, c_0, \ldots, c_{n-1}$ satisfy Conditions (1)–(3). Conditions (1) and (3) are immediate. To establish (2), we argue that θ^* witnesses that $\varphi^*(x, c_0, \ldots, c_{n-1}) \in \mathcal{ID}$ provably over B. Visibly, $\theta^* \in \mathrm{tp}(c_0, \ldots, c_{n-1}/B)$. Choose $c'_0, \ldots, c'_{n-1}$ realizing θ^*. On one hand, since $\psi \in \mathcal{ID}$ and $\theta \vdash \psi$, $\theta(c'_0, \ldots, c'_{n-1}, y_n) \in \mathcal{ID}$. On the other hand, by our choice of θ, $\varphi(x, c'_0, \ldots, c'_{n-1}, d) \in \mathcal{ID}$ for any d such that $\theta(c'_0, \ldots, c'_{n-1}, d)$ holds. Thus, $\varphi^*(x, c'_0, \ldots, c'_{n-1}) \in \mathcal{ID}$ since $\mathcal{ID}$ is a regular ideal.

Proposition 2.6 *If* $\mathrm{stp}(a/A)$ *is $\mathcal{ID}$-internal, then* $\mathrm{tp}(a/A)$ *is $\mathcal{ID}$-small.*

Proof. Choose $B \supseteq A$ independent from a over A, a B-definable formula $\varphi(x, \bar{y})$, and a tuple of elements $\bar{c}$ such that each $\mathrm{tp}(c/B)$ is $\mathcal{ID}$-small for each $c \in \bar{c}$, $\varphi(a, \bar{c})$ holds, and $\exists^{=1} x \varphi(x, \bar{c})$. But the formula $\varphi(x, \bar{c}) \in \mathcal{ID}$ provably over B via the formula $\exists^{=1} x \varphi(x, \bar{y})$, so $\mathrm{tp}(a/B)$ is $\mathcal{ID}$-small by Lemma 2.5. That $\mathrm{tp}(a/A)$ is $\mathcal{ID}$-small follows from Lemma 2.2.

The reader is cautioned that while $\mathcal{ID}$-internal types are $\mathcal{ID}$-small, this result does not extend to $\mathcal{ID}$-analyzable types. In fact, the theory and type mentioned in Remark 8.1.6 of [10] gives rise to an example of this. Much of the motivation of this section, and in particular how it differs from treatments in [3] and [10], revolves around how we handle $\mathcal{ID}$-analyzable types that are not $\mathcal{ID}$-small.

Definition 2.7 A strong type p is *foreign to $\mathcal{ID}$,* written $p \perp \mathcal{ID}$, if $p \perp q$ for every $\mathcal{ID}$-small q.

Lemma 2.8 *The following are equivalent for any regular ideal $\mathcal{ID}$ and any strong type p:*

1. *$p \perp \mathcal{ID}$;*
2. *$p \perp q$ for every $\mathcal{ID}$-internal strong type q;*
3. *$p \perp q$ for every $\mathcal{ID}$-analyzable strong type q;*
4. *If $p = \mathrm{stp}(a/A)$ then there is no $a' \in \mathrm{dcl}(Aa)$ such that $\mathrm{tp}(a'/A)$ is $\mathcal{ID}$-small.*

Proof. (1) $\Rightarrow$ (2) follows immediately from Proposition 2.6. (2) $\Rightarrow$ (3) follows by induction on the length of the $\mathcal{ID}$-analysis, using the fact that $p \perp \mathrm{tp}(b/B)$ and $p \perp \mathrm{tp}(a/Bb)$ implies $p \perp \mathrm{tp}(ab/B)$. (3) $\Rightarrow$ (4) is trivial, and (4) $\Rightarrow$ (1) follows immediately from (say) Corollary 7.4.6 of [10].

The reader is cautioned that when the regular ideal is not closed under $\mathcal{ID}$-analyzability, the following definitions differ from those in [10].

Definition 2.9 A partial type Γ is *$\mathcal{ID}$-large* if it is not $\mathcal{ID}$-small. Γ is *$\mathcal{ID}$-minimal* if it is $\mathcal{ID}$-large, but any forking extension of Γ is $\mathcal{ID}$-small. Γ is *$\mathcal{ID}_\perp$-minimal* if it is $\mathcal{ID}$-large, but any forking extension $\Gamma \cup \{\theta(x, c)\}$ is $\mathcal{ID}$-small whenever $\mathrm{stp}(c/\mathrm{dom}(\Gamma)) \perp \mathcal{ID}$.

Clearly $\mathcal{ID}$-minimality implies $\mathcal{ID}_\perp$-minimality, but one of the applications in Section 4 will use $\mathcal{ID}_\perp$-minimal types that are not $\mathcal{ID}$-minimal.

Lemma 2.10 *Let $\mathcal{ID}$ be any regular ideal. If a strong type p is both $\mathcal{ID}_\perp$-minimal and foreign to $\mathcal{ID}$, then p is regular.*

Proof. The point is that a counterexample to the regularity of p can be found within the set of realizations of p. If M is a-saturated and $p = \mathrm{tp}(a/M)$ is not regular then there are a tuple $\bar{c} = \langle c_1, \ldots, c_n \rangle$ realizing $p^{(n)}$ for some n and a realization b of p such that $\mathrm{tp}(a/M\bar{c})$ forks over M, $\mathrm{tp}(b/M\bar{c})$ does not fork over M, and $\mathrm{tp}(b/M\bar{c}a)$ forks over $M\bar{c}$. Let $q = \mathrm{tp}(a/M\bar{c})$ and choose an $L(M)$-formula $\theta(x, \bar{c}) \in q$ such that $p \cup \{\theta(x, \bar{c})\}$ forks over M. As $p \perp \mathcal{ID}$, $p^{(n)} \perp \mathcal{ID}$, so the $\mathcal{ID}_\perp$-minimality of p implies $\mathrm{tp}(a/M\bar{c})$ is $\mathcal{ID}$-small.

But, since p is foreign to $\mathcal{ID}$, $\mathrm{tp}(b/M\bar{c})$, which is a nonforking extension of p would be orthogonal to q by Lemma 2.8(2). In particular, $\mathrm{tp}(b/M\bar{c}a)$ would not fork over $M\bar{c}$.

The following easy 'transfer result' will be used in the subsequent sections.

Lemma 2.11 *Assume that B is algebraically closed, $p = \mathrm{tp}(a/B)$ is foreign to $\mathcal{ID}$, $q = \mathrm{tp}(b/B)$, and $b \in \mathrm{acl}(Ba) \setminus B$. Then q is foreign to $\mathcal{ID}$. If, in addition, p is $\mathcal{ID}$-minimal ($\mathcal{ID}_\perp$-minimal) then q is $\mathcal{ID}$-minimal ($\mathcal{ID}_\perp$-minimal) as well.*

Proof. If q were not foreign to $\mathcal{ID}$, then by Lemma 2.8(4) there is $c \in \mathrm{dcl}(Bb) \setminus B$ such that $\mathrm{tp}(c/B)$ is $\mathcal{ID}$-small. Since $\mathrm{tp}(c/B)$ is not algebraic it is not orthogonal to p, which, via Lemma 2.8(2), contradicts p being foreign to $\mathcal{ID}$. Thus $q \perp \mathcal{ID}$.

Next, suppose that p is $\mathcal{ID}$-minimal. Since $p \not\perp q$ and $p \perp \mathcal{ID}$, q cannot be $\mathcal{ID}$-small. To see that q is $\mathcal{ID}$-minimal, choose $C \supseteq B$ such that $\mathrm{tp}(b/C)$ forks over B. Then $\mathrm{tp}(a/C)$ forks over B, so $\mathrm{tp}(a/C)$ is $\mathcal{ID}$-small. Thus $\mathrm{tp}(b/C)$ is $\mathcal{ID}$-small by Lemma 2.5.

3 Chains and witnessing groups

Throughout this section $\mathcal{ID}$ always denotes a regular ideal.

Definition 3.1 We say A *is an $\mathcal{ID}$-subset of B,* written $A \subseteq_{\mathcal{ID}} B$, if $A \subseteq B$ and $\mathrm{stp}(b/A) \perp \mathcal{ID}$ for every finite tuple b from B. When M and N are models we write $M \preceq_{\mathcal{ID}} N$ when both $M \preceq N$ and $M \subseteq_{\mathcal{ID}} N$. A set A is *$\mathcal{ID}$-full* if $A \subseteq_{\mathcal{ID}} M$ for some (equivalently for every) a-prime model M over A.

Lemma 3.2 *Let $\mathcal{ID}$ be any regular ideal and assume M is a-saturated.*

1. *If $M \preceq N$ are models then $M \preceq_{\mathcal{ID}} N$ if and only if $\varphi(N) = \varphi(M)$ for all $\varphi \in L(M) \cap \mathcal{ID}$.*

2. *If $M \subseteq_{\mathcal{ID}} A$, then $M \preceq_{\mathcal{ID}} M[A]$, where $M[A]$ is any a-prime model over $M \cup A$.*

Proof. (1) First suppose $M \preceq_{\mathcal{ID}} N$ and choose $\varphi \in L(M) \cap \mathcal{ID}$. If $c \in \varphi(N)$ then $\mathrm{tp}(c/N)$ is $\mathcal{ID}$-small. If $\mathrm{tp}(c/M)$ were not algebraic, it would be nonorthogonal to an $\mathcal{ID}$-small type, contradicting $\mathrm{tp}(c/M) \perp \mathcal{ID}$. So $\mathrm{tp}(c/M)$ is algebraic, hence $c \in \varphi(M)$. Conversely, if there were $c \in N$ such that $\mathrm{tp}(c/M) \not\perp \mathcal{ID}$, then by Lemma 2.8(4) there is $c' \in \mathrm{dcl}(Mc) \setminus M$ such that $\mathrm{tp}(c'/M)$ is $\mathcal{ID}$-small. Then $\varphi(N) \neq \varphi(M)$ for any $\varphi \in \mathrm{tp}(c'/M) \cap \mathcal{ID}$.

(2) Recall that because M is a-saturated, $M[A]$ is dominated by A over M. Choose any tuple c from $M[A]$. If $\mathrm{tp}(c/M)$ were not foreign to $\mathcal{ID}$, then as M is a-saturated, there is an $\mathcal{ID}$-small type $q \in S(M)$ such that $\mathrm{tp}(c/M) \not\perp q$, hence $\mathrm{tp}(c/M)$ is not almost orthogonal to q. Since c is dominated by A over M, there is a from A such that $\mathrm{tp}(a/M)$ is not almost orthogonal to q, which contradicts $M \subseteq_{\mathcal{ID}} A$.

Definition 3.3 A *saturated chain* is an elementary chain $\langle M_\alpha : \alpha < \delta \rangle$ of a-saturated models in which $M_{\alpha+1}$ realizes every complete type over M_α for each $\alpha < \delta$. An *$\mathcal{ID}$-chain* is a sequence $\langle M_\alpha : \alpha < \delta \rangle$ of a-saturated models such that $M_\alpha \preceq_{\mathcal{ID}} M_\beta$ for all $\alpha < \beta < \delta$ and $M_{\alpha+1}$ realizes every type over M_α foreign to $\mathcal{ID}$. A chain (of either kind) is *$\mathcal{ID}$-full* if the union $\bigcup_{\alpha<\delta} M_\alpha$ is an $\mathcal{ID}$-full set.

In general, a saturated chain need not be $\mathcal{ID}$-full. However, if $\mathcal{ID}$ is either the ideal of algebraic formulas or superstable formulas (both of which are regular), then any a-saturated chain is $\mathcal{ID}$-full, since types are based on finite sets. A more complete explanation of this is given in the proof of

Lemma 4.2. By contrast, the following Lemma demonstrates that $\mathcal{ID}$-chains are always $\mathcal{ID}$-full.

Lemma 3.4 *Every $\mathcal{ID}$-chain is full. That is, if $\langle M_\alpha : \alpha < \delta \rangle$ is an $\mathcal{ID}$-chain, δ is a nonzero limit ordinal, and M_δ is a-prime over $\bigcup_{\alpha<\delta} M_\alpha$, then $M_\alpha \preceq_{\mathcal{ID}} M_\delta$ for all $\alpha < \delta$.*

Proof. By the characterization of $M \preceq_{\mathcal{ID}} N$ given by Lemma 3.2(1), the first sentence follows from the second. So fix an $\mathcal{ID}$-chain $\langle M_\alpha : \alpha < \delta \rangle$. Let $N = \bigcup_{\alpha<\delta} M_\alpha$ and let M_δ be a-prime over N. Fix any $\alpha < \delta$. Since $M_\alpha \subseteq_{\mathcal{ID}} M_\beta$ for all $\alpha < \beta < \delta$, $M_\alpha \subseteq_{\mathcal{ID}} N$, so $M_\alpha \preceq_{\mathcal{ID}} M_\delta$ by Lemma 3.2(2).

Definition 3.5 A formula θ is *weakly $\mathcal{ID}$-minimal (weakly $\mathcal{ID}_\perp$-minimal)* if $\{\theta\}$ is $\mathcal{ID}$-minimal ($\mathcal{ID}_\perp$-minimal).

We now state offer two complementary propositions. The main point of both is that they produce regular types that are 'close' to a given regular ideal. The advantage of (1) is that one obtains $\mathcal{ID}$-minimality at the cost of requiring the chain to be $\mathcal{ID}$-full. In (2) the fullness condition is automatically satisfied by Lemma 3.4, but one only gets $\mathcal{ID}_\perp$-minimality.

Proposition 3.6 *Fix a regular ideal $\mathcal{ID}$, a countable, stable theory T, and an $\mathcal{ID}$-large formula φ.*

1. **Either** *there is a weakly $\mathcal{ID}$-minimal formula $\psi \vdash \varphi$* **or** *for every $\mathcal{ID}$-full saturated chain $\langle M_n : n \in \omega \rangle$ with $\varphi \in L(M_0)$, there is an $\aleph_1$-isolated, $\mathcal{ID}$-minimal $p \in S(\bigcup_n M_n)$ with $\varphi \in p$ and $p \perp \mathcal{ID}$.*

2. **Either** *there is a weakly $\mathcal{ID}_\perp$-minimal formula $\psi \vdash \varphi$* **or** *for every $\mathcal{ID}$-chain $\langle M_n : n \in \omega \rangle$ with $\varphi \in L(M_0)$, there is an $\aleph_1$-isolated, $\mathcal{ID}_\perp$-minimal $p \in S(\bigcup_n M_n)$ with $\varphi \in p$ and $p \perp \mathcal{ID}$.*

Moreover, in either of the two 'second cases' the type p is regular.

Proof. (1) Assume that there is no weakly $\mathcal{ID}$-minimal $\psi \vdash \varphi$. Fix an $\mathcal{ID}$-full saturated chain $\langle M_n : n \in \omega \rangle$ with $\varphi \in L(M_0)$, let $N = \bigcup_{n\in\omega} M_n$, and let M_ω be $\aleph_1$-prime over N. Let $\Delta_0 \subseteq \Delta_1 \subseteq \ldots$ be finite sets of formulas with $L = \bigcup_{n\in\omega} \Delta_n$. We inductively construct a sequence $\langle \varphi_n : n \in \omega\rangle$ of $\mathcal{ID}$-large formulas as follows: Let φ_0 be our given φ. Given $\varphi_n \vdash \varphi_0$ that is an $\mathcal{ID}$-large $L(M_n)$-formula

$$A_n = \{\psi \in L(M_{n+1}) : \psi \vdash \varphi_n,\ \psi \text{ is } \mathcal{ID}\text{-large and forks over } M_n\}.$$

As M_{n+1} realizes every type over M_n foreign to $\mathcal{ID}$ and φ_n is not weakly $\mathcal{ID}$-minimal, A_n is nonempty. Choose $\varphi_{n+1} \in A_n$ so as to minimize $R(\psi, \Delta_n, 2)$. Let $\Gamma = \{\varphi_n : n \in \omega\}$. We first argue that Γ has a unique extension to a complete type in $S(N)$.

Claim. $\Gamma \vdash \neg\psi(x, b)$ for all $\psi(x, b) \in \mathcal{ID} \cap L(N)$.

Proof. If the Claim were to fail, then $\Gamma \cup \{\psi(x, b)\}$ would be consistent, hence would be realized in M_ω, say by an element c. As the chain is $\mathcal{ID}$-full, $c \in N$. For any n such that $c \in M_n$, φ_{n+1} was chosen to fork over M_n, yet is realized in M_n, which is impossible.

Now let $\psi(x, b)$ be any $L(N)$-formula. Choose n such that $\psi(x, y) \in \Delta_n$. As φ_{n+1} was chosen to be of minimal $\mathrm{R}(-,\Delta_n, 2)$-rank, it is not possible for both $\varphi_{n+1} \wedge \psi(x, b)$ and $\varphi_{n+1} \wedge \neg\psi(x, b)$ to be in A_n. For definiteness, suppose $\varphi_{n+1} \wedge \psi(x, b) \notin A_n$. Since φ_{n+1} forks over M_n, it must be that $\varphi_{n+1} \wedge \psi(x, b) \in \mathcal{ID}$. Since $\varphi_{n+1} \in \Gamma$, the Claim implies that $\Gamma \vdash \neg\psi(x, b)$. Thus Γ implies a complete type in $S(N)$, which we call p.

By construction p is $\aleph_1$-isolated and is $\mathcal{ID}$-large by the Claim. Since M_ω is $\aleph_1$-saturated and p is $\aleph_1$-isolated, there is a realization c of p in M_ω. If p were not foreign to $\mathcal{ID}$ then by Lemma 2.8(4) there would be $c' \in \mathrm{dcl}(Nc)\setminus N$ with c'/N $\mathcal{ID}$-small, directly contradicting $\mathcal{ID}$-fullness.

It remains to show that any forking extension of p is $\mathcal{ID}$-small. Let $\theta(x, a^*)$ be any $L(\mathfrak{C})$-formula such that $p \cup \theta(x, a^*)$ forks over N. Then for some n, $\theta(x, y) \in \Delta_n$ and $\varphi_{n+1} \wedge \theta(x, a^*)$ forks over N. Thus, $R(\varphi_{n+1} \wedge \theta(x, a^*), \Delta_n, 2) < R(\varphi_{n+1}, \Delta_n, 2)$. Let e list the parameters occurring in φ_{n+1}. Since M_{n+1} is saturated, choose $a' \in M_{n+1}$ such that $\mathrm{tp}(ea') = \mathrm{tp}(ea^*)$. It follows from the minimality of $R(\varphi_{n+1}, \Delta_n, 2)$ that $\varphi_{n+1}\wedge\theta(x, a') \notin A_n$, hence $\varphi_{n+1} \wedge \theta(x, a') \in \mathcal{ID}$. Thus, $\varphi_{n+1} \wedge \theta(x, a^*) \in \mathcal{ID}$ by the invariance of $\mathcal{ID}$.

As for (2) assume that there is no $\mathcal{ID}_\perp$-minimal formula implying φ. Choose an $\mathcal{ID}$-chain $\langle M_n : n \in \omega\rangle$, which is automatically $\mathcal{ID}$-full by Lemma 3.4. The construction of Γ and p are analogous, taking $A'_n = \{\psi(x, b) \in A_n : \mathrm{tp}(b/M_n) \perp \mathcal{ID}\}$ in place of A_n at each step. All that is affected is the final paragraph. As we only need to establish $\mathcal{ID}_\perp$-minimality, choose a formula $\theta(x, a^*)$ with $\mathrm{tp}(a^*/N) \perp \mathcal{ID}$. Choose n as above such that, in addition, $\mathrm{tp}(a^*/N)$ is based and stationary on some finite $b \in M_n$. Choose $a' \in M_{n+1}$ such that $\mathrm{tp}(bea') = \mathrm{tp}(bea^*)$ and continue as above.

In both cases, the regularity of p follows immediately from Lemma 2.10.

Recall that a stable theory has *NDIDIP* if for every elementary chain $\langle M_n : n \in \omega \rangle$ of models, every type that is nonorthogonal to some a-prime model over $\bigcup_{n\in\omega} M_n$ is nonorthogonal to some M_n. Relationships between NDIDIP and NDOP are explored in [8].

Proposition 3.7 *Fix a countable, stable theory T with NDIDIP and a regular ideal $\mathcal{ID}$ such that the formula '$x = x$'$\notin \mathcal{ID}$.*

1. *If there is an an $\mathcal{ID}$-full, saturated chain $\langle M_n : n \in \omega \rangle$, but there is no weakly $\mathcal{ID}$-minimal formula then there is an abelian group witness to unsuperstability, where in addition the generic type of the intersection is both $\mathcal{ID}$-minimal and foreign to $\mathcal{ID}$.*

2. *If there is no weakly $\mathcal{ID}_\perp$-minimal formula then there is an abelian group witness to unsuperstability where the generic type of the intersection is $\mathcal{ID}_\perp$-minimal and foreign to $\mathcal{ID}$.*

Proof. (1) Fix an $\mathcal{ID}$-full, saturated chain $\langle M_n : n \in \omega \rangle$ and let $N = \bigcup_{n\in\omega} M_n$. Using Proposition 3.6(1) choose $p \in S(N)$ to be $\aleph_1$-isolated, foreign to $\mathcal{ID}$, and $\mathcal{ID}$-minimal, hence regular. Since T has NDIDIP, $p \not\perp M_n$. Since p is regular and M_n is a-saturated, by Claim X 1.4 of [12] there is a regular type $r_0 \in S(M_n)$ nonorthogonal to p. Let r denote the nonforking extension of r_0 to N. As p and r are nonorthogonal there is an integer m such that $p^{(m)}$ is not almost orthogonal to $r^{(\omega)}$. Since p is $\aleph_1$-isolated and M_n is a-saturated, Na is dominated by N over M_n for any a realizing p. Thus $p^{(1)}$ is not almost orthogonal to $r^{(\omega)}$ over N. Choose $k \geq 1$ maximal such that $p^{(k)}$ is almost orthogonal to $r^{(\omega)}$ over N and choose $\bar{c}$ realizing $p^{(k)}$. Let $B = \mathrm{acl}(N\bar{c})$ and choose a realization a of the nonforking extension of p to B.

By Theorem 1 of [4], there is $b \in \mathrm{dcl}(Ba) \setminus B$ and a type-definable, connected group A with a regular generic type q (so A is abelian by Poizat's theorem [11]) and a definable regular, transitive action of A on $p_1(\mathfrak{C})$, where

$p_1 = \text{tp}(b/B)$. By Lemma 2.11 the type p_1 and hence q are both foreign to $\mathcal{ID}$ and $\mathcal{ID}$-minimal. By Theorem 2 of [5] there is a definable supergroup $A_0 \supseteq A$. By an easy compactness argument we may assume A_0 is abelian as well. Furthermore, by iterating Theorem 2 of [5] we obtain a descending sequence $\langle A_n : n \in \omega \rangle$ of subgroups of A_0 with $A = \bigcap_{n \in \omega} A_n$.

Thus far we have not guaranteed that A_{n+1} has infinite index in A_n. In order to show that there is a subsequence of the A_n's with this property and thereby complete the proof of the Proposition, it suffices to prove the following claim:

Claim For every $n \in \omega$ there is $m \geq n$ such that $[A_n : A_m]$ is infinite.

Proof. By symmetry it suffices to show this for $n = 0$. Assume that this were not the case, i.e., that $[A_0, A_m]$ is finite for each m. Then A has bounded index in A_0. We will obtain a contradiction by showing that the definable set A_0 is weakly $\mathcal{ID}$-minimal. First, since q is $\mathcal{ID}$-large, the formula defining A_0 is $\mathcal{ID}$-large as well. Let $\varphi(x, e)$ be any forking extension of the formula defining A_0 and let $E \subseteq A_0$ be the set of realizations of $\varphi(x, e)$. Let $\{C_i : i < 2^\kappa \leq 2^{\aleph_0}\}$ enumerate the cosets of A contained in A_0. For each i, $E \cap C_i$ is a forking extension of C_i. Since every C_i is a translate of A whose generic type is $\mathcal{ID}$-minimal, this implies that $E \cap C_i$ is $\mathcal{ID}$-small for each i. Hence, $\varphi(x, e) \in \mathcal{ID}$ by compactness (and the fact that $\mathcal{ID}$ is an ideal). Thus, the formula defining A_0 is weakly $\mathcal{ID}$-minimal, contradiction.

The proof of (2) is identical, choosing an $\mathcal{ID}$-chain satisfying the hypotheses and using Proposition 3.6(2) in place of 3.6(1).

4 Applications

Our first application gives a 'trichotomy' for strictly stable theories in a countable language. It uses the ideal of superstable formulas. Let $\mathbf{R}^\infty$ denote the ideal of

Definition 4.1 $\mathbf{R}^\infty$ denotes the ideal of superstable formulas (i.e., all formulas φ with $R^\infty(\varphi) < \infty$).

Equivalently, $\varphi \in \mathbf{R}^\infty$ if and only if for all cardinals $\kappa \geq 2^{|T|}$, for any model M of size κ containing the parameters of φ, there are at most κ complete types over M extending φ. In a sense, the following Lemma generalizes the fact that in a superstable theory, any union of a chain of a-saturated models is a-saturated.

Lemma 4.2 $\mathbf{R}^\infty$ *is a regular ideal, any elementary chain* $\langle M_n : n \in \omega \rangle$ *of a-saturated models is* $\mathbf{R}^\infty$*-full, and there are no weakly* $\mathbf{R}^\infty$*-minimal formulas.*

Proof. Invariance under automorphisms of $\mathfrak{C}$ is clear and $\mathbf{R}^\infty$ being an ideal follows by counting types. To show regularity, choose $\psi(y) \in \mathbf{R}^\infty$ and $\theta(x, y) \in L(\mathfrak{C})$ such that $\theta(x, b) \in \mathbf{R}^\infty$ for every b realizing ψ. Choose $\kappa \geq 2^{|T|}$ and a model M of size κ containing the hidden parameters of both ψ and θ. Then there are at most κ types $p(x, y) \in S(M)$ extending $\theta(x, y) \wedge \psi(y)$, so the projection $\exists y(\theta(x, y) \wedge \psi(y)) \in \mathbf{R}^\infty$ as only κ types $q(x) \in S(M)$ extend it.

To establish fullness, fix an elementary chain $\langle M_n : n \in \omega \rangle$ of a-saturated models. Let $N = \bigcup_{n \in \omega} M_n$ and choose an a-prime model M_ω over N. Because of Lemma 2.8(4), in order to show that $N \subseteq_{\mathcal{ID}} M_\omega$ it suffices to show that no element of $c \in M_\omega \setminus N$ is $\mathbf{R}^\infty$-small. So choose $c \in M_\omega$ such that $\mathrm{tp}(c/N)$ is $\mathcal{ID}$-small and we will show that $c \in N$. On one hand, since $\mathrm{tp}(c/N)$ contains a superstable formula there is a finite n such that $\mathrm{tp}(c/N)$ is based on M_n. On the other hand, since M_ω is a-prime over N, $\mathrm{tp}(c/N)$ is a-isolated. Thus $\mathrm{tp}(c/M_n)$ is a-isolated as well (see e.g., Theorem IV 4.3(1) of [12]). Since M_n is a-saturated, this implies $c \in M_n \subseteq N$.

To show that there are no weakly $\mathbf{R}^\infty$-minimal formulas, suppose that a formula φ has the property that any forking extension of φ is $\mathbf{R}^\infty$-small. We will show that $\varphi \in \mathbf{R}^\infty$ by counting types. Fix a cardinal $\kappa \geq 2^{|T|}$ and a model M of size κ containing the parameters of φ. Let $M_0 \preceq M$ have size $|T|$ that also contains the parameters containing φ. It suffices to show that every $p \in S(M_0)$ extending φ has at most κ extensions to types in $S(M)$. Clearly, there is a unique nonforking extension of p and any forking extension of p contains an $L(M)$-formula witnessing the forking. Each such forking formula $\psi \in \mathbf{R}^\infty$, so there are at most κ $q \in S(M)$ extending ψ. Since the total number of $\psi \in L(M)$ is at most κ, p has at most κ extensions to types in $S(M)$.

Theorem 4.3 *Let T be a stable, unsuperstable theory in a countable language. Then at least one of the following three conditions occurs:*

1. *T has the dimensional order property (DOP); or*

2. *T has NDOP, but is deep (i.e., there is a sequence $\langle M_n : n \in \omega \rangle$ such that $\mathrm{tp}(M_{n+1}/M_n) \perp M_{n-1}$ for all $n \geq 1$); or*

3. *There is an abelian group witness to unsuperstability (see Definition 1.1) in which the generic type of the intersection is both $\mathbf{R}^\infty$-minimal and foreign to $\mathbf{R}^\infty$.*

Proof. To begin, Corollary 1.12 of [8] asserts that any such theory T has NDIDIP. Since T is not superstable the formula '$x = x$'$\notin \mathbf{R}^\infty$. As well, by Lemma 4.2 there are no weakly $\mathbf{R}^\infty$-minimal formulas, so Proposition 3.7(1) asserts that an abelian group witness to unsuperstability exists, whose generic type is regular and both $\mathbf{R}^\infty$-minimal and foreign to $\mathbf{R}^\infty$.

Our second application comes from an attempt to solve the 'Main Gap for $\aleph_1$-saturated models.' As in the previous theorem, the relevant setting is where a countable theory T is stable, unsuperstable, with NDOP, and is shallow. The main open question is whether, for such a theory every nonalgebraic type r is nonorthogonal to a regular type. The following result sheds some light on this issue. In order to analyze this problem, fix a nonalgebraic, stationary type r over the empty set. Let

$$\mathcal{ID}_r = \{\varphi \in L(\mathfrak{C}) : r \perp \varphi\}$$

Verifying that $\mathcal{ID}_r$ is an invariant ideal is straightforward. To see that it is a regular ideal, fix $L(\mathfrak{C})$-formulas $\psi(y) \in \mathcal{ID}_r$ and $\theta(x, y)$ such that $\theta(x, b) \in \mathcal{ID}_r$ for every b realizing ψ. Choose an a-saturated model M containing the parameters of ψ and θ, pick a realization c of the nonforking extension of r to M, and let $M[c]$ be any a-prime model over Mc. To show that $\varphi(x) := \exists y(\theta(x, y) \wedge \psi(y)) \perp r$ it suffices to prove that any realization of φ in $M[c]$ is contained in M. So choose any $a \in \varphi(M[c])$. Choose $b \in M[c]$ such that $\theta(a, b) \wedge \psi(b)$ holds. Since $r \perp \psi$, $b \in M$. But then $\theta(x, b)$ is over M and is $\perp r$, so $a \in M$ as well. Thus $\mathcal{ID}_r$ is a regular ideal.

Theorem 4.4 *Assume that a countable theory T is stable, unsuperstable, has NDOP, and is shallow. If a nonalgebraic, stationary type r is orthogonal to every regular type, then there is an abelian group witness to unsuperstability in which the generic type of the intersection $A = \bigcap_n A_n$ is both $(\mathcal{ID}_r)_\perp$-minimal and foreign to $\mathcal{ID}_r$.*

Proof. Fix such a type r. By naming constants we may assume that r is over the empty set. Note that any formula $\varphi \in r$ is not an element of $\mathcal{ID}_r$, so '$x = x$'$\notin \mathcal{ID}_r$.

Claim. There is no weakly $(\mathcal{ID}_r)_\perp$-minimal formula.

Proof. Assume that φ were $(\mathcal{ID}_r)_\perp$-minimal. We construct a regular type $p \not\perp r$ as follows: Choose an a-saturated model M containing the parameters in φ, pick a realization c of the nonforking extension of r to M, and choose an a-prime model $M[c]$ over Mc. Since φ is $\mathcal{ID}_r$-large we can find an $a \in M[c] \setminus M$ realizing φ. Choose such an a and let $p = \text{tp}(a/M)$. Clearly, $p \not\perp r$. To see that p is regular, first note that p is $(\mathcal{ID}_r)$-minimal since p is $\mathcal{ID}_r$-large and extends φ. As well, p is foreign to $\mathcal{ID}_r$, since if it were not, then by Lemma 2.8(4) there would be $b \in \text{dcl}(Ma)$ with $\text{tp}(b/M)$ $\mathcal{ID}_r$-small. But then $tp(c/Mb)$ would fork over M, implying that r is nonorthogonal to an $\mathcal{ID}_r$-small type, which is a contradiction. So p is $(\mathcal{ID}_r)$-minimal and foreign to $\mathcal{ID}_r$, hence is regular by Lemma 2.10.

The theorem now follows immediately from Proposition 3.6(2).

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