An old friend revisited: Countable models of ω-stable theories

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Abstract

We work in the context of ω-stable theories. We obtain a natural, algebraic equivalent of ENI-NDOP and discuss recent joint proofs with S. Shelah that if an ω-stable theory has either ENI-DOP or is ENI-NDOP and is ENI-deep, then the set of models of T with universe ω is Borel complete.

In 1983 Shelah, Harrington, and Makkai proved Vaught's conjecture for ω-stable theories [11]. In that paper they determined which ω-stable theories have fewer than $2^{\aleph_0}$ countable models and proved a strong structure theorem for models of such a theory. As in most verifications of Vaught's conjecture for specific classes, little attention was paid to countable models of ω-stable theories have 'many' models. It is curious that following the publication of [11] in 1984, the investigation of the class of countable models of an arbitrary ω-stable theory lay fallow for many years.[1]

One explanation for this hiatus may have been a lack of test questions. How could one describe the complexity of a class of countable structures beyond asserting that there are $2^{\aleph_0}$ nonisomorphic ones? A remedy was provided by the collective works of Becker, Kechris, Hjorth, Friedman, Stanley,

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[1]We understand that recently Martin Koerwien has been working independently on similar problems.

and others (see e.g.[1, 3, 5]) who, building on earlier work of Vaught [12], developed the concept of the *Borel complexity* of a class of countable structures. Whereas the full technology is much more general, we focus on a special case. For a given (countable) vocabulary τ, we concentrate on the Polish space $\mathcal{S}(\tau)$ of τ-structures with universe ω and subspaces thereof.[2] Call a subspace K of $\mathcal{S}(\tau)$ *invariant* if K is closed under isomorphism. It is well known that any invariant set K can be viewed as the set of models with universe ω of some $L_{\omega_1,\omega}$-sentence φ in the vocabulary τ. If K and K' are invariant sets, possibly in different vocabularies, we say that K is *Borel reducible* to K' if there is a Borel function from K to K' such that

$$\mathcal{A} \cong \mathcal{B} \quad \text{if and only if} \quad f(\mathcal{A}) \cong f(\mathcal{B})$$

for all $\mathcal{A}, \mathcal{B} \in K$. An invariant K is *Borel complete* if every invariant K' is Borel reducible to it. We call a theory T *Borel complete* if the set of models of T with universe ω is Borel complete.

It is easily seen that the set of graphs (either symmetric or directed) with universe ω is Borel complete. Somewhat more surprisingly, Friedman and Stanley [3] proved that the set of subtrees of ${}^{<\omega}\omega$ is Borel complete. This paper presents some recent results of Saharon Shelah and the author that identify certain classes of ω-stable theories as being Borel complete. It should be noted that there a number of open questions remain in this area. While we verify that some classes of ω-stable theories are Borel complete, we do not have a full characterization. Also, it is easy to see that if T is Borel complete, then its class of countable models has unbounded Scott heights in ω_1. At present we do not know whether the converse holds for ω-stable theories.

We set the stage by recalling three facts about ω-stable theories:

- Prime models exist over arbitrary sets A. They are unique up to isomorphism over A and are atomic over A.

- Types over models are based and stationary over finite subsets. That is, for any $p \in S(M)$ there is a finite $A \subseteq M$ such that p is the unique nonforking extension in $S(M)$ of the restriction $p|A$.

[2]To aid clarity, in certain places we shall freely replace ω by another fixed countable universe, e.g., ω^2 or ${}^{<\omega}\omega$.

- Strongly regular types are ubiquitous and are well-behaved. In particular, if M is a model and $p \not\perp M$, then $p \not\perp q$ for some strongly regular $q \in S(M)$. Moreover, if $q, r \in S(M)$ are both strongly regular and nonorthogonal, then $\dim(q, N) = \dim(r, N)$ for any N extending M.

Our approach is to modify definitions occurring in Shelah's 'top down analysis' of superstable theories to distinguish between classes of *countable* models. The main difference is that there is no cardinal gap between 'infinite' and $\aleph_0$. Thus, for example, if a theory is strong enough to require that the dimension of a certain regular type be infinite in any model of the theory, then it is futile to use its dimension to distinguish between nonisomorphic countable models of the theory. This can have a drastic impact on the complexity of the models of a theory. An extreme example is the 'standard checkerboard example' of an ω-stable theory having the dimensional order property (DOP). It has the maximal number of uncountable models (as does any stable theory with DOP) but is actually $\aleph_0$-categorical.

The fundamental modifications all appear in [11] but we develop them in the general setting of ω-stable theories *without* restricting to those having few countable models. In this instance, rather than looking at all strongly regular types over a model, they suggested identifying those that are 'eventually non-isolated'. Such types can have finite dimension in a countable model, so specifying the dimension of such a type gives positive information. More precisely, call a complete type $p \in S(M)$ ENI if p is strongly regular and there is a finite $A \subseteq M$ on which p is based, stationary, and nonisolated. In [11] they suggested a variant of DOP, called ENI-DOP, which had a technical definition, but was just what was needed to translate Shelah's original proofs that 'DOP implies complexity' to the context of countable models. We now see that the definition (or more precisely its negation ENI-NDOP) can be stated much more naturally in an algebraic context. Call three models $\{M_0, M_1, M_2\}$ an *independent triple of models* if $M_0 = M_1 \cap M_2$ and $M_1 \underset{M_0}{\perp} M_2$.

Definition 1 An ω-stable theory T has *ENI-NDOP* if the prime model over any independent triple of ω-saturated models is ω-saturated. We say T has *ENI-DOP* if it fails to have ENI-NDOP.

That is, an ω-stable theory T has ENI-NDOP if and only if the a-prime model over any independent triple of a-models is atomic over the triple.[3] Phrased in this way, it is insightful to compare this property with the status of NOTOP (the negation of the omitting types order property) in the superstable setting: In [9] Shelah proves that a superstable theory with NDOP satisfies NOTOP if and only if the a-prime model over an independent triple of a-models is atomic over the triple. Thus, in the ω-stable context, this is precisely ENI-NDOP. As well, it is useful to note that a routine Downward Löwenheim-Skolem argument shows that it is equivalent to restrict to countable models. Thus, an ω-stable theory T has ENI-NDOP if and only if the prime model over any independent triple of countable, saturated models is saturated.

Before continuing, let us prove that this definition is the equivalent to the more technical version appearing in [11].

Proposition 2 *An ω-stable theory T has ENI-DOP if and only if there is an independent triple $\{M_0, M_1, M_2\}$ of ω-saturated models, a model N prime over M_1M_2, and an ENI type $p \in S(N)$ such that $p \perp M_1$ and $p \perp M_2$.*

Proof. (Sketch) First, assume that $\{M_0, M_1, M_2\}$ is an independent triple of ω-saturated models for which the prime model N over M_1M_2 is not ω-saturated. Choose $p \in S(N)$ to be the nonalgebraic type of smallest Morley rank such that there is a finite $A \subseteq N$ on which p is based and stationary and $p|A$ is omitted in N. That p is strongly regular follows from the minimality condition. Since $p|A$ is omitted in N it is surely nonisolated; hence p is ENI. If $p \not\perp M_i$ for some $i \in \{1, 2\}$, then choose a strongly regular $q \in S(M_i)$ such that $p \not\perp q$. Choose a finite $B \subseteq M_i$ on which q is based and stationary, and let $N_0 \preceq N$ be prime over AB. Let $p', q' \in S(N_0)$ be types parallel to p and q, respectively. Now $\dim(p', N) = \dim(q', N) = \omega$, where the first equality follows from $p \not\perp q$ (see e.g., [2]) and the second equality follows from the ω-saturation of M_i. But this contradicts $p|A$ being omitted in N.

Conversely, suppose that $\{M_0, M_1, M_2\}$ is any independent triple of models, N is any prime model over M_1M_2, and $p \in S(N)$ is an ENI type orthogonal to both M_1 and M_2. We find a finite subset $B^* \subseteq N$ for which $p|B^*$ is omitted in N. First, choose a finite $B \subseteq N$ on which p is based and

[3] In an ω-stable theory the a-models are precisely the ω-saturated models.

stationary and $p|B$ is not isolated. Choose finite sets $A_1 \subseteq M_1$ and $A_2 \subseteq M_2$ such that taking $B^* = BA_1A_2$, we have

$$B^* \underset{A_1A_2}{\perp} M_1M_2 \quad \text{and} \quad B^* \underset{B^* \cap M_0}{\perp} M_0$$

A computation similar to the proof of $(c) \Rightarrow (d)$ in Lemma X, 2.2 of [9] shows that $p|B^* \vdash p|B^*M_1M_2$. Since $p|B^*$ is not isolated, cB^* is not atomic over M_1M_2 for any c realizing $p|B^*$ (hence $p|B^*M_1M_2$). Thus $p|B^*$ is omitted in N. ∎

Examples of ω-stable theories with ENI-DOP include differentially closed fields (see e.g., Marker's [8] in this volume) and a variant of the standard checkerboard example: Let $L = \{U, R, f_n\}_{n \in \omega}$ and let T guarantee that any $M \models T$ satisfies

- $R(a, b, c) \rightarrow (U(a) \wedge U(b) \wedge \neg U(c))$;
- Each $f_n : U(M)^2 \rightarrow \neg U(M)$;
- $\{R(a, b, z) : (a, b) \in U(M)^2\}$ forms a partition of $\neg U(M)$; and
- If $n \neq m$ and $(a, b) \in U(M)^2$, $R(a, b, f_n(a, b))$ and $f_n(a, b) \neq f_m(a, b)$.

If $\{M_0, M_1, M_2\}$ is an independent triple of saturated models, $a \in U(M_1) \setminus M_0$, and $b \in U(M_2) \setminus M_0$, then the type $p(a, b, z) = \{R(a, b, z)\} \cup \{z \neq f_n(a, b) : n \in \omega\}$ is omitted in any atomic model over $M_1 \cup M_2$, so T has ENI-DOP.

Our definition of ENI-DOP makes the following Theorem conceptually easy:

Theorem 3 *If T is ω-stable with ENI-DOP, then T is Borel complete.*

Proof. (Sketch) Suppose T is ω-stable with ENI-DOP. It is an easy exercise in coding to show that the class of countable bipartite graphs is Borel complete, so it suffices to find a Borel reduction from this class into the class of countable models of T. By the comment following Definition 1, choose an independent triple $\{M, N, Q\}$ of *countable* saturated models of T such that the prime model $\mathcal{M}^*$ over $N \cup Q$ is not saturated. Choose a type $p \in S(\mathcal{M}^*)$ of minimal Morley rank that has a finite subset $A \subseteq \mathcal{M}^*$ on which p is based and stationary, yet $p|A$ is omitted in $\mathcal{M}^*$. The minimality of rank ensures that p is strongly regular, hence ENI. Choose an independent set $\{N_i : i \in \omega\} \cup \{Q_j : j \in \omega\}$ over M where $\mathrm{tp}(N_i/M) = \mathrm{tp}(N/M)$ and $\mathrm{tp}(Q_j/M) = \mathrm{tp}(Q/M)$ for all $i, j \in \omega$. For each pair $(i, j) \in \omega^2$ $\mathrm{tp}(N_iQ_j/M) = \mathrm{tp}(NQ/M)$ so there is an automorphism $\sigma_{i,j}$ of the monster satisfying $\sigma_{i,j}(N) = N_i$, $\sigma_{i,j}(Q) = Q_j$, and $\sigma_{i,j} = id$ on M. Let $\mathcal{M}^*_{i,j} = \sigma_{i,j}(\mathcal{M}^*)$, let $p_{i,j}$ be the corresponding conjugate of p, and let $\mathcal{N}_0$ be prime over $\bigcup_{i,j} \mathcal{M}^*_{i,j}$.

Now suppose that we are given a bipartite graph $\mathcal{G} = (\omega^2, E_G)$. Define a model $\mathcal{N}_\mathcal{G} = \bigcup_n \mathcal{N}_n$ of T, where $\mathcal{N}_0$ is as above and, given $\mathcal{N}_n$ let $I_n = \{a_{i,j} : (i, j) \in E_\mathcal{G}\}$ be an independent set over $\mathcal{N}_n$, where each $a_{i,j}$ is a realization of $p_{i,j}|\mathcal{N}_n$ and choose $\mathcal{N}_{n+1}$ to be prime over $\mathcal{N}_n \cup I_n$. The model $\mathcal{N}_0$ does not depend on the graph $\mathcal{G}$, but every $\mathcal{N}_n$ for $n \geq 1$ does. Since T is ω-stable the isolated types over any set are dense and the prime model over any set is constructible. Thus, for each n, given an enumeration of $N_n \cup I_n$ and an enumeration of the $L(N_nI_n)$-formulas, the atomic diagram of N_{n+1} is determined. It follows via coding that the mapping $\mathcal{G} \mapsto \mathcal{N}_\mathcal{G}$ can be made to be Borel. For any pair (i, j), our construction yields that

$$\dim(p_{i,j}, \mathcal{N}_\mathcal{G}) = \begin{cases} \omega & \text{if } (i,j) \in E_\mathcal{G} \\ 0 & \text{if } (i,j) \notin E_\mathcal{G} \end{cases}$$

Furthermore, it is easily checked that if $\mathcal{G} \cong \mathcal{H}$, then $\mathcal{N}_G \cong \mathcal{N}_\mathcal{H}$. Determining when nonisomorphism is preserved is more challenging. If $p_{i,j}$ is based and stationary on $(N_i \cup Q_j) \setminus M$, then nonisomorphism will indeed be preserved. However, in the general case, $p_{i,j}$ might depend on parameters from M as well. The 'patch' is to define a coarser relation on the space of bipartite graphs. Namely, we say $\mathcal{G} \sim \mathcal{H}$ if and only if $G \setminus F_G \cong H \setminus F_H$ for some finite subsets $F_G \subseteq G$ and $F_H \subseteq H$. We prove that the space of bipartite graphs remains Borel complete with respect to the relation $\sim$, and that the mapping above satisfies $\mathcal{N}_\mathcal{G} \cong \mathcal{N}_\mathcal{H}$ implies $\mathcal{G} \sim \mathcal{H}$. It follows that T is Borel complete. ∎

Thus, we may restrict our attention to ω-stable theories with ENI-NDOP. Although [11] concentrates on theories with few countable models, it is already implicit in [11] that any countable model of such a theory admits a tree decomposition. We pause to make these notions precise. Throughout, a *tree* is a nonempty, downward closed subset of ${}^{<\omega}\omega$. For $\eta \neq \langle\rangle$, η^- denotes the immediate predecessor of η.

Definition 4 Fix M any model. A *partial decomposition* $\mathcal{D}$ *of* M is a set of pairs $\mathcal{D} = \{(M_\eta, a_\eta) : \eta \in \mathcal{T}^{\mathcal{D}}\}$ indexed by a tree $\mathcal{T}^{\mathcal{D}}$ satisfying:

1. $M_{\langle\rangle}$ is an atomic substructure of M and $\{a_\nu : lg(\nu) = 1\}$ is a maximal independent over $M_{\langle\rangle}$ set of realizations of strongly regular types $q_\nu \in S(M_{\langle\rangle})$;

2. For each nonempty $\eta \in \mathcal{T}$, M_η is atomic over $M_{\eta^-} \cup \{a_\eta\}$ and $\{a_\nu : \nu$ an immediate successor of $\eta\}$ is a maximal independent over M_η set of realizations of strongly regular types $q_\nu \in S(M_\eta)$ satisfying $q_\nu \perp M_{\eta^-}$.

A *decomposition* of M is a partial decomposition such that M is prime over $\bigcup\{M_\eta : \eta \in \mathcal{T}\}$.

Note that there is no restriction placed on $a_{\langle\rangle}$. It is included to minimize the complexity of the definition. There is a natural partial order on partial decompositions of M, namely $\mathcal{D}_1 \leq \mathcal{D}_2$ if and only if $\mathcal{T}^{\mathcal{D}_1}$ is a subtree of $\mathcal{T}^{\mathcal{D}_2}$ and $(M_\eta^{\mathcal{D}_1}, a_\eta^{\mathcal{D}_1}) = (M_\eta^{\mathcal{D}_2}, a_\eta^{\mathcal{D}_2})$ for each $\eta \in \mathcal{T}^{\mathcal{D}_1}$, which gives rise to the notion of a 'maximal' partial decomposition.

As noted above the following theorem really only uses ideas present in [11], which in turn follow from ideas in Chapter XI of [9].

Theorem 5 *Suppose T is ω-stable with ENI-NDOP. Then every countable model $M \models T$ has a decomposition. Moreover, every maximal partial decomposition of M is a decomposition of M.*

One has tremendous flexibility in choosing a decomposition of a given model M of such a theory. One can freely choose any atomic submodel for $M_{\langle\rangle}$. Next, there are several choices of maximal independent sequences of realizations of strongly regular types over $M_{\langle\rangle}$. Then, for each a_ν with $lg(\nu) = 1$ one can freely choose an atomic model over $M_{\langle\rangle} \cup \{a_\nu\}$, etc. While it is true that atomic submodels of M over a given set are isomorphic over the set, this does not make them unique. It is an excellent exercise for the reader to see the vast freedom one has in constructing maximal decompositions of the countable saturated model of the theory in Example 11.

This example suggests that if we want the complexity of the decomposition of a model M to reflect the complexity of the isomorphism type of M, we should restrict our freedom in choosing a decomposition. One natural way to do this is to insist that at each 'choice' we take a *maximal* atomic submodel over the requisite set. This leads to a better decomposition result.

Definition 6 Fix a model M. A *partial ENI decomposition* of M is a partial decomposition of M in which $\text{tp}(a_\eta/M_{\eta^-})$ is ENI for every nonempty $\eta \in \mathcal{T}$. An *ENI decomposition* of M is a partial ENI decomposition of M where M is prime over $\bigcup\{M_\eta : \eta \in \mathcal{T}\}$.

The following theorem is proved in [6], but is likely known to others.

Theorem 7 *Suppose M is a countable model of an ω-stable theory with ENI-NDOP. Then:*

1. *Any partial decomposition of M in which every M_η is chosen to be* maximal atomic *over the requisite set is a partial ENI decomposition (hence maximal partial ENI decompositions of M exist); and*

2. *Any maximal partial ENI decomposition of M is an ENI decomposition of M.*

The reader is cautioned that even ENI decompositions of a model need not be unique. In fact, even if the ENI depth is finite countable models of T can have ENI decompositions of differing ENI depths. Despite this, Theorem 10 below demonstrates that if the theory T admits a countable model with an ENI-decomposition indexed by a nonwellfounded tree, then the class of countable models of T is Borel complete.

It turns out, however, that if one is seeking a dividing line, having non-wellfounded ENI-decompositions is too restrictive. Example 13, which is kind of a hybrid of Examples 11 and 12, has a bound on the complexity of ENI-decompositions, yet the class of countable models allows for coding of arbitrary trees.

Definition 8 A *chain* is a finite sequence $\langle (M_i, p_i) : i \leq k \rangle$ such that $M_0 \preceq M_1 \preceq \ldots M_k$ and for every $i < k$, M_i is countable, $p_i \in S(M_i)$ is regular, M_{i+1} is prime over M_i and a realization of p_i, and (when $i > 0$) $p_i \perp M_{i-1}$. An *ENI-chain* is a chain, where in addition each p_i is an ENI type.

That is, a chain is a potential 'branch' of a decomposition tree of a model and an ENI-chain is a branch of an ENI-decomposition tree. In terms of Borel complexity, sharper results are obtained via the following hybrid notion that encapsulates the essence of Example 13.

A type $p \in S(M)$ is *supportive* (of an ENI-type) if there is a chain $\langle (M_i, p_i) : i \leq k \rangle$ with $M_0 = M$, $p_0 = p$, and *some* p_i is ENI. As well, a *supportive chain* is a chain in which every p_i is supportive. Thus, ENI-chains are supportive, but not conversely. In Example 13 every ENI-chain occurring in a decomposition is short, while there are arbitrarily long supportive chains in the decompositions of certain models.

Definition 9 Fix an ω-stable theory T with ENI-NDOP. T is *deep* if there is an ω-sequence in which every proper initial segment is a chain. (This is consistent with Shelah's notion of a deep theory in [9].) T is *ENI-deep* if there is an ω-sequence in which every proper initial segment is an ENI-chain and T is $e-deep$ if there is such an ω-sequence with each proper initial segment a supportive chain.

As every ENI-chain is supportive, an ENI-deep theory is necessarily e-deep. But the theory in Example 13 is e-deep but not ENI-deep. In [6] we succeed in proving the following Theorem.

Theorem 10 *If T is ω-stable, has ENI-NDOP and is e-deep, then T is Borel complete.*

The proof of Theorem 10 is rather involved. At first blush, once one knows Friedman and Stanley's theorem that countable trees are Borel complete, it seems like the proof should be easy. One can fix a sequence $\mathcal{S} = \langle (M_i, p_i) : i < \omega \rangle$ such that every proper initial segment is a supportive chain. It is easy to get a Borel mapping $\mathcal{T} \mapsto M_{\mathcal{T}}$ from the set of subtrees of ${}^{<\omega}\omega$ to the class of countable models of T having a decomposition tree indexed by $\mathcal{T}$ in which every branch is isomorphic to an initial segment of $\mathcal{S}$. Moreover, almost any reasonable way of doing this will preserve isomorphism, i.e., if $\mathcal{T} \cong \mathcal{T}'$ then $M_{\mathcal{T}} \cong M_{\mathcal{T}'}$. However, the nonuniqueness of decompositions prevents one from immediately asserting that nonisomorphism is preserved. The solution is twofold. First, given a tree $\mathcal{T}$, one 'pads' $\mathcal{T}$, i.e., exhibits a Borel mapping $\mathcal{T} \mapsto \overline{\mathcal{T}}$, where $\overline{\mathcal{T}}$ consists of 'many copies' of $\mathcal{T}$. Then, by adapting many of the arguments of [10] and the method of quasi-isomorphisms described in [4] (but allowing only finitely many exceptions instead of countably many) we show that the composition map $\mathcal{T} \mapsto \overline{\mathcal{T}} \mapsto M_{\overline{\mathcal{T}}}$ preserves nonisomorphism.

One other point is worth making. Whereas we are able to prove Theorem 10, our arguments are rather crude. The 'real' question of determining whether two decompositions are sufficiently different as to imply nonisomorphism of the models they generate remains open. This issue would need to be addressed by someone looking at e-shallow theories (i.e., not e-deep) and attempting to determine the precise Borel complexity of the class.

We close by giving the three examples alluded to above.

Example 11 *The theory of a unary function without loops.*

Let $L = \{f\}$, where f is a unary function symbol and the theory T assert that every element has infinitely many preimages and that there are 'no loops' i.e., $\forall x(f^{(n)}(x) \neq x)$ for all $n \geq 1$. It is easily checked that T is ω-stable with ENI-NDOP.

If M is any model of T and $a \in M$, define the *component* of a in M to be

$$C(a) = \{b \in M : f^{(n)}(b) = f^{(m)}(a) \text{ for some } n, m \in \omega\}$$

It is easily checked that any two components of a model M are disjoint or equal, and if M is countable then any two components are isomorphic. Thus, the isomorphism type of a countable model is determined by the number of components. In particular, T has only countably many nonisomorphic countable models.

However, T is deep. To see this, note that if $N \preceq M$, then a nonalgebraic type $p \in S(M)$ is orthogonal to N if and only if $p \vdash f^{(n)}(x) = a$ for some $a \in M$ and $n \geq 1$, but $p \vdash f^{(m)}(x) \notin N$ for every $m \in \omega$. Using this characterization it is easy to construct an ω-chain $M_0 \preceq M_1 \preceq \ldots$ where $\text{tp}(M_{n+1}/M_n) \perp M_{n-1}$ for all $n \geq 1$ witnessing that T is deep. The reason why deepness does not imply many models is that none of the relevant types in such a witness are ENI, hence all relevant dimensions (other than the number of components) are necessarily $\aleph_0$ in any countable model of T. More precisely, for a given model M there is a unique nonisolated complete 1-type, namely the type specifying that x is in a component disjoint from M. ∎

In the next example we add additional structure to make the requisite types ENI.

Example 12 *A unary function with no loops having ENI preimages.*

Let $L = \{f, S\}$, where f and S are both unary function symbols. The theory T asserts that f is as in Example 11, S is a '**Z**-like successor function' i.e., every element has an immediate S-successor and an immediate S-predecessor, and $S^{(n)}(x) \neq x$ for all $n \geq 1$. Furthermore, for any element a of any model, $f^{-1}(a)$ is closed under S, i.e., $\forall x f(S(x)) = f(x)$.

This theory is again ω-stable with ENI-NDOP, but it is also ENI-deep. Indeed, the characterization of which types over M are orthogonal to N when $N \preceq M$ is identical to the one given in Example 11. However, in this case any chain of models witnessing that T is deep simultaneously witnesses that T is ENI-deep. Thus, T is Borel complete by Theorem 10. ∎

Our final example illustrates the distinction between ENI-chains and supportive chains.

Example 13 *An ω-stable theory T with ENI-NDOP that is e-deep but not ENI-deep.*

Let $L = \{P, Q, f, S\}$, where P and Q are unary relations dividing the universe into two sorts. f is a unary function symbol acting on the P-part as a unary function with no loops as in the two preceding examples. Additionally, f describes an infinite-to-one surjection of Q onto P. So for each element x in the P-sort, $f^{-1}(x)$ has infinite intersection with both P and Q, By contrast $f^{-1}(y) = \emptyset$ for any element y of the Q-sort. Finally, S is a **Z**-like successor function on the Q-part satisfying

$$\forall y[Q(y) \rightarrow f(S(y)) = f(y)]$$

The theory T is ω-stable, ENI-NDOP, e-deep, but not ENI-deep. It is illustrative to see how arbitrary subtrees of ${}^{<\omega}\omega$ can be coded into countable models of T. Let M_0 denote the prime model of T. The isomorphism type of M_0 can be described by specifying that $P(M_0)$ consists of a single component (hence is isomorphic to the prime model of the theory in Example 11) such that, in addition, for every $a \in P(M_0)$, $f^{-1}(a) \cap Q(M_0)$ consists of a single **Z**-chain. Let a^* denote an arbitrary element of $P(M_0)$. Recursively define a (Borel) injection

$$h : {}^{<\omega}\omega \rightarrow P(M_0)$$

to guarantee that $h(\langle\rangle) = a^*$ and that h induces a bijection between $\{\eta\hat{\ }\langle n\rangle : n \in \omega\}$ and $f^{-1}(h(\eta)) \cap P(M_0)$ for every $\eta \in {}^{<\omega}\omega$.

Now, given an arbitrary tree $\mathcal{T} \subseteq {}^{<\omega}\omega$, we form a countable model $M_\mathcal{T}$ using the function h and **Z**-chains in the Q-sort as 'markers.' Specifically, given such a $\mathcal{T}$ let $M_\mathcal{T}$ be the (elementary) extension of M_0 formed by adding exactly one extra **Z**-copy to $f^{-1}(h(\eta)) \cap Q$ for each $\eta \in \mathcal{T}$. A moment's thought shows that the mapping $\mathcal{T} \mapsto M_\mathcal{T}$ is Borel and preserves both isomorphism and nonisomorphism, hence T is Borel complete. ∎

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