MODEL COMPLETENESS FOR TRIVIAL, UNCOUNTABLY CATEGORICAL THEORIES OF MORLEY RANK 1

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1. Introduction

The present paper is a direct continuation of [2], where it is shown that any strongly minimal trivial theory is model complete after naming constants for a model. In this paper we show that this result may be generalized to any uncountably categorical, trivial theory of Morley Rank 1. Specifically we show:

Theorem 1. *If T is a trivial uncountably categorical theory of Morley Rank 1 then $T(M_0)$, the theory obtained from T by naming constants for a model, is model complete.*

We use this theorem to derive the same corollaries for the theories covered by the theorem as were derived for the strongly minimal case in [2]. We also note that the theorem is in some senses optimal. Specifically we can easily construct trivial Morley Rank 1 theories which are not categorical and for which the conclusion of the theorem fails. Also Marker in [3] constructs trivial totally categorical theories of Morley Rank 2 which are not model complete after naming any set of constants.

Throughout the ensuing sections we rely heavily on the exposition presented in [2], so some familiarity with this paper will help the reader follow the present work. We will use basic concepts from stability theory without comment, see [1] for background material in the subject. Since we are generally concerned with the situation where we have models $M \subseteq N$ and are attempting to prove that $M \preceq N$ we do not have the luxury of working in a universal domain $\mathfrak{C}$ and assuming that all models are elementary substructures of this structure. This entails that some of our notation and terminology differs somewhat from most references in stability theory, namely we must be very careful to specify the ambient model for some of the notions. We establish the following two notational conventions to fix the meaning of some basic stability theoretic concepts in our context. In the ensuing we assume T is a stable theory.

Notation 1. *For $M_0 \preceq M$ models of T and $\overline{a}, \overline{b}$ from M we write $\overline{a} \downarrow_{M_0} \overline{b}$ in M to mean that $tp_M(\overline{a}/M_0\overline{b})$ does not fork over M_0. We note that in this context $\overline{a} \downarrow_{M_0} \overline{b}$*

The second author was partially supported by NSF grant DMS-0300080.

in M if and only if for any formula $\phi(\overline{x},\overline{b}) \in tp_M(\overline{a}/M_0\overline{b})$ there is $\overline{c} \in M_0$ such that $M \models \phi(\overline{c},\overline{b})$.

The reader is cautioned that when $M_0 \preceq M$, $M_0 \preceq N$, $M \subseteq N$ and $\overline{a},\overline{b}$ are from M $\overline{a} \perp_{M_0} \overline{b}$ in M does not necessarily imply that $\overline{a} \perp_{M_0} \overline{b}$ in N.

Notation 2. *If M_0 is any model of T, $p \in S(M_0)$ is any type, and $\phi(\overline{x},\overline{y})$ is any M_0-definable formula, then $d_p\overline{x}\phi(\overline{x},\overline{y})$ is the definition of the ϕ-part of p, namely of the set $\{\overline{b} \subseteq M_0 : \phi(\overline{x},\overline{b}) \in p\}$.*

It follows by results from stability theory that $d_p\overline{x}\phi(\overline{x},\overline{y})$ is equivalent to a positive boolean combination of instances $\phi(\overline{c},\overline{y})$ of $\phi(\overline{x},\overline{y})$ for $\overline{c}$ in M_0. We also readily see that for any M such that $M_0 \preceq M$, any $\overline{a},\overline{b}$ in M such that $\overline{a} \models p$ and $\overline{a} \perp_{M_0} \overline{b}$ in M, and any formula $\phi(\overline{x},\overline{y})$ with parameters from M_0, $M \models \phi(\overline{a},\overline{b})$ if and only if $M \models d_p\overline{x}\phi(\overline{x},\overline{b})$. For details on this see [4].

2. Canonical Amalgamation of Trivial Rank 1 Structures

This preliminary section establishes an amalgamation result for trivial theories of Morley Rank 1 which will be essential in the proof of the main technical Proposition proved in the next section.

We make the following assumptions: T is a complete trivial theory of Morley Rank 1 in a language $\mathcal{L}$ and M_0 is a model of T.

Lemma 1. *Suppose $M_0 \preceq M_1$, $M_0 \preceq M_2$, and $M_2 \cap M_1 = M_0$. Then there is a unique $\mathcal{L}$-structure N with universe $M_1 \cup M_2$ such that $M_1 \preceq N$ and $M_2 \preceq N$.*

Proof: We first show that we may assume there is a large highly saturated $\mathfrak{C} \models T$ so that $M_1 \prec \mathfrak{C}$ and $M_2 \prec \mathfrak{C}$. Pick $\mathfrak{C}$ to be any large saturated model of $T(M_0)$, we may assume without loss of generality that $M_1 \prec \mathfrak{C}$. Next note that since M_0 is algebraically closed in $\mathfrak{C}$, if $\overline{a} \subset M_1 \setminus M_0$ then $tp_{\mathfrak{C}}(\overline{a}/M_0)$ is non-algebraic. Hence there is an $\mathcal{L}(M_0)$-elementary embedding of M_2 into $\mathfrak{C}$ so that $g \upharpoonright M_0 = Id$ and $g(M_2) \cap M_1 = M_0$. By replacing M_2 by $g(M_2)$ we may asusme that $M_2 \prec \mathfrak{C}$. Also since we have that $M_1 \cap M_2 = M_0$ by triviality we must have that $M_1 \perp_{M_0} M_2$ in $\mathfrak{C}$.

Claim 1. The set $M_1 \cup M_2$ is the universe of a substructure of $\mathfrak{C}$.

Proof: Let $t(\overline{x},\overline{y})$ be any $\mathcal{L}(M_0)$-term. Suppose that $\overline{a} \subset M_1$, $\overline{b} \subset M_2 \setminus M_0$, and $c \in \mathfrak{C}$ such that $\mathfrak{C} \models c = t(\overline{a},\overline{b})$. Then $\mathfrak{C} \models c \in acl(M_0\overline{a}\overline{b})$. If $\mathfrak{C} \models c \in acl(M_0\overline{a})$ then $c \in M_1$ since $M_1 \preceq \mathfrak{C}$. If $\mathfrak{C} \models c \notin acl(M_0\overline{a})$ then by triviality $\mathfrak{C} \models c \in acl(\overline{b})$ and so $c \in M_2$ since $M_2 \preceq \mathfrak{C}$. Hence $M_1 \cup M_2$ is a substructure of $\mathfrak{C}$.

Let $N \subset \mathfrak{C}$ denote the substructure in Claim 1.

Note that for each $\phi(\overline{x},\overline{y}) \in \mathcal{L}(M_0)$ quantifier free, $\overline{a} \subset M_1$, and $\overline{b} \subset M_2 \setminus M_0$, $N \models \phi(\overline{a},\overline{b})$ if and only if $d_p\overline{x}\phi(\overline{x},\overline{y}) \in tp_{M_2}(\overline{b}/M_0)$, where $p = tp_{M_1}(\overline{a}/M_0)$.

Claim 2. $N \models T(M_0)$, $M_1 \preceq N$, and $M_2 \preceq N$.

Proof: All three of these facts will follow from the assertion that $N \preceq \mathfrak{C}$. For this choose $\phi(z, \overline{x}, \overline{y}) \in \mathcal{L}(M_0)$, $\overline{a} \subset M_1$, and $\overline{b} \subset M_2 \setminus M_0$. Assume that $\mathfrak{C} \models \exists z \phi(z, \overline{a}, \overline{b})$. If $\mathfrak{C} \models \phi(z, \overline{a}, \overline{b})$ is algebraic then $N \models \exists z \phi(z, \overline{a}, \overline{b})$ exactly as in Claim 1. Otherwise $\phi(\mathfrak{C}, \overline{a}, \overline{b})$ is infinite, in particular we may find $c' \mathop{\smile\!\!\!\!\!\!\,|}_{M_0} \overline{a}\overline{b}$ realizing $\phi(z, \overline{a}, \overline{b})$. Thus since $M_0 \preceq \mathfrak{C}$ and types not forking over M_0 are finitely satisfiable in M_0 there is $c \in M_0$ such that $\mathfrak{C} \models \phi(c, \overline{a}, \overline{b})$. Hence we have that $N \preceq \mathfrak{C}$.

Finally we need to establish the uniqueness statement. Suppose that N' has universe $M_1 \cup M_2$, $M_1 \preceq N'$, and $M_2 \preceq N'$. By the above observation it suffices to show that if $\overline{a} \subset M_1$, $\overline{b} \subset M_2 \setminus M_0$, and $\phi(\overline{x}, \overline{y}) \in \mathcal{L}(M_0)$ quantifier free then $N' \models \phi(\overline{a}, \overline{b})$ if and only if $d_p \overline{x} \phi(\overline{x}, \overline{y}) \in tp_{M_2}(\overline{b}/M_0)$ where $p = tp_{M_1}(\overline{a}/M_0)$. Hence we may choose an elementary embedding f of N into $\mathfrak{C}$ and verify this condition for the image of N under f. For simplicity we assume that $N' \preceq \mathfrak{C}$. Fix a quantifier free $\mathcal{L}(M_0)$-formula $\phi(\overline{x}, \overline{y})$, $\overline{a} \subset M_1$, $\overline{b} \subset M_2 \setminus M_1$. Let $p = tp_{M_1}(\overline{a}/M_0)$ and $q = tp_{M_2}(\overline{b}/M_0)$. Now suppose that $\mathfrak{C} \models \phi(\overline{a}, \overline{b})$. Once again note that we must have that $M_1 \mathop{\smile\!\!\!\!\!\!\,|}_{M_0} M_2$ in $\mathfrak{C}$ and hence that $\mathfrak{C} \models d_p \overline{x} \phi(\overline{x}, \overline{b})$. Since $M_2 \preceq N' \preceq \mathfrak{C}$ we get that $M_2 \models d_p \overline{x} \phi(\overline{x}, \overline{b})$, i.e. $d_p \overline{x} \phi(\overline{x}, \overline{y}) \in tp_{M_2}(\overline{b}/M_0)$. This yields that $N \models \phi(\overline{a}, \overline{b})$. Hence we get that the identity map on $M_1 \cup M_2$ is an isomorphism between N and N'. $\square$

Definition 1. For models M_1, M_2 as above we will refer to the model N obtained from the Lemma as the *canonical expansion* of M_1 and M_2, and we will simply denote it by $M_1 M_2$.

Lemma 2. *If $M, N \models T(M_0)$, $M \subseteq N$, $q \in S(M_0)$, $\overline{a} \subset M$, and $R \in \mathcal{L}(M_0)$ is quantifier free then: if $d_q \overline{y} R(\overline{x}, \overline{y}) \in tp_M(\overline{a}/M_0)$ then $d_q \overline{y} R(\overline{x}, \overline{y}) \in tp_N(\overline{a}/M_0)$.*

Proof: Notice that without loss of generality we may assume that $\overline{a} \subset M \setminus M_0$. Because of our assumptions on T and the saturation of M_0 we may find $M_1 \preceq M_0$ such that M_1 contains any parameters from M_0 appearing in R, q is the non-forking extension of $q' = q \restriction M_1$, and such that $\langle M_0, c \rangle_{c \in M_1}$ is a saturated $\mathcal{L}(M_1)$-structure. Choose $\overline{b}^* \in M_0$ realizing q'. Suppose that $M \models d_q \overline{y} R(\overline{a}, \overline{y})$, then $M \models d_{q'} \overline{y} R(\overline{a}, \overline{y})$ and hence $M \models R(\overline{a}, \overline{b}^*)$ since we have that $\overline{a} \mathop{\smile\!\!\!\!\!\!\,|}_{M_1} \overline{b}^*$ in M. Since $M \subseteq N$ and R is quantifier free $N \models R(\overline{a}, \overline{b}^*)$. Notice that since $M_0 \preceq N$ we get that $tp_N(b^*/M_1) = q'$ and $\overline{a} \mathop{\smile\!\!\!\!\!\!\,|}_{M_1} \overline{b}^*$ in N. These two fact yield that $N \models d_{q'} \overline{y} R(\overline{a}, \overline{y})$ and thus $N \models d_q \overline{y} R(\overline{a}, \overline{y})$. $\square$

Corollary 1. *Suppose that M_1, N_1, M_2 are all models of $T(M_0)$. Furthermore suppose that $M_1 \subseteq N_1$ and $N_1 \cap M_2 = M_0$. Then the canonical expansion of M_1 and M_2 is a substructure of the canonical expansion of N_1 and M_2.*

Proof: Choose $R(\overline{x},\overline{y}) \in \mathcal{L}(M_0)$ quantifier free, $\overline{a} \subset M_1$, and $\overline{b} \subset M_2 \setminus M_0$. Let $q = tp_{M_2}(\overline{b}/M_0) = tp_{M_1M_2}(\overline{b}/M_0) = tp_{N_1M_2}(\overline{b}/M_0)$. Note these equalities follow immediately from Lemma 1. Suppose that $M_1M_2 \models R(\overline{a},\overline{b})$, then $d_q\overline{y}R(\overline{x},\overline{y}) \in tp_{M_1M_2}(\overline{a}/M_0)$. Since $M_1 \preceq M_1M_2$, we have that $d_q\overline{y}R(\overline{x},\overline{y}) \in tp_{M_1}(\overline{a}/M_0)$. By the previous Lemma we get that $d_q\overline{y}R(\overline{x},\overline{y}) \in tp_{N_1}(\overline{a}/M_0)$. Since $N_1 \preceq N_1M_2$ we must have that $d_q\overline{y}R(\overline{x},\overline{y}) \in tp_{N_1M_2}(\overline{a}/M_0)$. Hence $N_1M_2 \models R(\overline{a},\overline{b})$. □

Corollary 2. *Suppose $M_1 \subseteq N_1$, $M_2 \subseteq N_2$, and $N_1 \cap N_2 = M_1 \cap M_2 = M_0$. Then M_1M_2 is a substructure of N_1N_2.*

Proof: Apply the previous corollary twice, first to establish that $M_1M_2 \subseteq N_1M_2$, and secondly to get that $N_1M_2 \subseteq N_1N_2$. This gives the desired result. □

We finish with some examples to show that our assumptions in this section are necessary.

First of all notice that Lemma 1 fails if the theory is not trivial. In particular $Th(\mathbb{Q},+)$, which is strongly minimal and locally modular, witnesses this. Secondly the Morley Rank 1 assumption is necessary. Consider the theory T in a language $\mathcal{L}$ consisting of a single unary predicate U and a unary function f where $M \models T$ if and only if $U(M)$ is infinite and co-infinite and f defines a bijection between the unordered pairs in $U(M)$ and the singletons in $\neg U(M)$. One readily sees that this theory is trivial, uncountably categorical, and of Morley Rank 2 but that Lemma 1 fails for it. Finally it is reasonable to assume that Lemma 1 should hold with M_0 replaced by $acl(\emptyset)$, but the following example demonstrates that we need the pair M_1 and M_2 in Lemma 1 to be disjoint over a model.

Example 1. We begin with a theory T' in a language $\mathcal{L}'$ consisting of two unary functions f, g. We let T' by the theory of the structure with universe $\mathbb{Z}^2$ where

$$f(\langle m,n\rangle) = \langle m+1,n\rangle \text{ and } g(\langle m,n\rangle) = \langle m,n+1\rangle.$$

This theory is trivial and strongly minimal and notice that $acl(\emptyset) = \emptyset$. Next in any model of T' we have a definable equivalence relation Ex_1x_2, y_1y_2 on pairs given by:

$$(x_1 = y_1 \wedge x_2 = y_2) \vee (f(x_1) = x_2 \wedge f(y_1) = y_2) \vee (g(x_1) = x_2 \wedge g(y_1) = y_2).$$

Now let $\mathcal{L} = \{E\}$ and let T be the theory of the reduct of models of T' to E. Note that T is strongly minimal and trivial and we have that $acl(\emptyset) = \emptyset$. Finally notice that if M and N are disjoint models of T (hence disjoint over $acl(\emptyset)$) then there are two distinct models with universe $M \cup N$ such that both M and N are elementary substructures.

3. Mutually Algebraic Sequences

This section is devoted to establishing the main technical tool we will need in order to prove the main theorem. We begin with some assumptions.
In this section T is always a trivial uncountably categorical theory of Morley Rank 1, and M_0 is an uncountable model of T. We fix models $M \subseteq N$ of $T(M_0)$ such that M is of strictly greater cardinality than M_0.

We will need the following definitions.

Definition 2. A formula $\phi(\overline{x})$ is *absolute* if for any $\overline{c} \subset M$, $M \models \phi(\overline{c})$ if and only if $N \models \phi(\overline{c})$.

Definition 3. Let $R(\overline{z})$ be any $\mathcal{L}(M_0)$-formula. Fix $M \models T(M_0)$ and $\overline{c} \in M^n$. We write $M \models \overline{c}$ *is* R *mutually algebraic* if:

1. $M \models R(\overline{c})$.
2. For every partition $\overline{z} = \overline{x}^\frown\overline{y}$, if $\overline{c} = \overline{d}^\frown\overline{e}$ then $M \models \neg d_p\overline{x}R(\overline{x}, \overline{e})$ for every $p \in S_{|\overline{x}|}(M_0)$.

Remark 1. As an easy consequence of the definition of mutually algebraic we get that for $\overline{c} \not\subseteq M_0$, if $M \models \overline{c}$ is R mutually algebraic where $\overline{c} = c_0...c_{n-1}$ and $c_i \notin M_0$ then $M \models c_0, c_i, c_1..., c_{n-1}$ is $\widetilde{R}$ mutually algebraic where:

$$\widetilde{R}(x_0, y, x_1...x_{n-1}) := R(x_0...x_{n-1}) \wedge y = x_i.$$

Lemma 3. *For fixed R there is a type $\Gamma_R(\overline{z}w)$ with parameters from M_0 such that for any $b \in M$ the set of realizations of $\Gamma_R(\overline{z}b)$ is the set:*

$$\{\overline{c} \in M^n : M \models \overline{c} \textit{ is } R \textit{ mutually algebraic and } c_{n-1} = b\}.$$

Proof: Simply let $\Gamma_R(\overline{z}w)$ be the following set of formulas:

$$\bigcup_{\overline{z}=\overline{x}^\frown\overline{y}} \{\neg d_p\overline{x}R(\overline{x}, \overline{y}) : p \in S_{|\overline{x}|}(M_0)\} \cup \{R(\overline{z}) \wedge z_{n-1} = w\}$$

□

Lemma 4. *Suppose $M \models T(M_0), b \in M \setminus M_0$ and that $R(\overline{z})$ is given.*

1. *If $\overline{c} \in M^n, c_{n-1} = b$, and $M \models \overline{c}$ is R mutually algebraic then $\overline{c} \subseteq acl(M_0b) \setminus M_0$.*
2. *$\{\overline{c} \in M^n : M \models \overline{c}$ is R mutually algebraic and $c_{n-1} = b$ $\}$ is finite.*

Proof: 1) Let $\overline{e}$ be the subsequence of $\overline{c}$ consisting of those elements in $acl(M_0b)$. Let $\overline{d}$ be such that $\overline{c} = \overline{d}^\frown\overline{e}$, note that without loss of generality we may assume that $\overline{e}$ is a final segment. Note that for every $d_i \in \overline{d}$ we have that $d_i \notin acl(M_0\overline{e})$. Hence since T is assumed to be trivial of Morley Rank 1 we have that $\overline{d} \perp_{M_0} \overline{e}$ in M. So, if $\overline{d} \neq \emptyset$

this contradicts that $M \models \overline{c}$ is R mutually algebraic. Which yields that $\overline{c} \subseteq acl(M_0 b)$. But notice there is nothing special about b so in fact we get that $\overline{c} \subseteq acl(M_0 c_i)$ for all $c_i \in \overline{c}$. So if for some c_i we have that $c_i \in M_0$ then $\overline{c} \subseteq M_0$ which contradicts that $b \notin M_0$. Hence $\overline{c} \subseteq acl(M_0 b) \setminus M_0$.

2) The set

$$\{\overline{c} \in M^n : M \models \overline{c} \text{ is } R \text{ mutually algebraic and } c_{n-1} = b\}$$

is type definable with parameters from $M_0 b$ by Lemma 3, and is a subset of $acl(M_0 b)$ by part 1. Hence it is finite by compactness. $\square$

We now need to make more assumptions about our theory. Fix an $\mathcal{L}(M_0)$ formula, $\phi(x, \overline{y})$ which is absolute between M and N and such that in any model of $T(M_0)$ the set defined by $\exists^{\leq r}\overline{y}\phi(x, \overline{y})$ is infinite and co-infinite. For notation let $A(x)$ be the set defined by $\exists^{\leq r}\overline{y}\phi(x, \overline{y})$ and $B(x)$ its complement.

Definition 4. For any $M \models T(M_0)$ and any $\sigma \in 2^n$ an n-tuple $\overline{c} \in M^n$ is *of type σ in M* if for all $i < n, M \models A(c_i)$ if and only if $\sigma(i) = 0$.

Notation 3. *For $\sigma \in 2^n$ we let $\sigma(x_0 ... x_{n-1})$ be the formula:*

$$\bigwedge_{\sigma(i)=0} A(x_i) \wedge \bigwedge_{\sigma(i)=1} B(x_i).$$

Also for $\overline{c} \in M^n$ we write $\sigma_M(\overline{c})$ for the unique $\sigma \in 2^n$ such that $M \models \sigma(\overline{c})$.

For convenience we also define:

Definition 5. *$\leq$ denotes the "componentwise partial order" on 2^n. Namely for $\sigma_1, \sigma_2 \in 2^n, \sigma_1 \leq \sigma_2$ if and only if for all $i < n, \sigma_1(i) \leq \sigma_2(i)$.*

Notice that if for some $a \in M$, $M \models \neg\exists^{\leq r}\overline{y}\phi(a, \overline{y})$ then $N \models \neg\exists^{\leq r}\overline{y}\phi(a, \overline{y})$. Hence we have:

Remark 2. If $M \subseteq N$ are models of $T(M_0)$ and $\overline{c} \in M^n$ then $\sigma_M(\overline{c}) \leq \sigma_N(\overline{c})$.

Notation 4. *For a formula R and $\sigma \in 2^n$ we will write $M \models \overline{c}$ is (R, σ) mutually algebraic in place of $M \models \overline{c}$ is $R \wedge \sigma$ mutually algebraic.*

We are now in a position to prove one of our main lemmas.

Lemma 5. *(Transfer) Suppose $M, N \models T(M_0)$, $M \subseteq N$, R is an absolute formula, $\sigma \in 2^n$, $\overline{c} \in M^n$ and $M \models \overline{c}$ is (R, σ) mutually algebraic. Then there is an absolute $\mathcal{L}(M_0)$ formula R^* and $\sigma^* \in 2^n$ such that $N \models \overline{c}$ is (R^*, σ^*) mutually algebraic.*

Proof: Fix $N, M, R, \sigma, \overline{c}$. Let $M_1 \preceq M_0$ be countable such that R and ϕ are formulas in $\mathcal{L}(M_1)$. First note that if $\overline{c} \subseteq M_0$ then the Lemma is immediate. So by Lemma 4 we may assume that $\overline{c} \subseteq M \setminus M_0$. For any proper partition $\overline{z} = \overline{x}^\frown\overline{y}$ (i.e. $lg(\overline{x}) \geq 1$) let $\overline{c} = \overline{d}^\frown\overline{e}$ be the associated partition of $\overline{c}$. Now choose $\overline{e}_i$ for $i < n$ in M_0 such that $tp(\overline{e}_i/M_1) = tp(\overline{e}/M_1)$ and such that the set $\{\overline{e}_0 ... \overline{e}_{n-1}\}$ is independent over M_1. Note this is possible since M_0 is saturated and uncountable while M_1 is countable. Let

$$R^*(\overline{z}) = R(\overline{z}) \wedge \bigwedge_{\overline{z}=\overline{x}^\frown\overline{y}} \big(\bigwedge_{i<n} \neg R(\overline{x}, \overline{e}_i) \big).$$

Since R is absolute so is R^*. Choose σ^* such that $N \models \sigma^*(\overline{c})$.

Claim. $N \models \overline{c}$ is (R^*, σ^*) mutually algebraic.

For the ensuing argument we assume that N is elementarily embedded in a large saturated model $\mathfrak{C}$ of T. Note of course that M need not be an elementary submodel of $\mathfrak{C}$.

We first show that $N \models R^*(\overline{c}) \wedge \sigma^*(\overline{c})$. Since R is absolute we clearly have $N \models R(\overline{c}) \wedge \sigma^*(\overline{c})$. Fix a partition $\overline{z} = \overline{x}^\frown\overline{y}$ and $i < n$ and suppose that by way of contradiction $N \models R(\overline{d}, \overline{e}_i)$ (where $\overline{c} = \overline{d}^\frown\overline{e}$).

Let $p = tp_N(\overline{d}/M_1)$. So $N \models d_p\overline{x}R(\overline{x}, \overline{e}_i)$, since $\overline{d} \subseteq N \setminus M_0$ and $\overline{e}_i \subseteq M_0$. But $tp_N(\overline{e}_i/M_1) = tp_N(\overline{e}/M_1)$ so we also have that $N \models d_p\overline{x}R(\overline{x}, \overline{e})$.

Now let N' be an isomorphic copy of N over M_0 elementarily embedded in $\mathfrak{C}$ such that $N' \downarrow_{M_0} N$ in $\mathfrak{C}$ and fix f an automorphism of $\mathfrak{C}$ fixing M_0 pointwise and taking N to N'. Let M' be the image of M under f. So we also have that M' is isomorphic to M over M_0. Also note that $M' \downarrow_{M_0} M$ in $\mathfrak{C}$. Let $N^* = NN'$ and let $M^* = MM'$, the canonical amalgamations. By Lemma 1 we have that $N, N' \preceq N^*$ and that $M, M' \preceq M^*$ and by Corollary 2 we have that $M^* \subseteq N^*$.

Let $\overline{d}^* = f(\overline{d})$. Since $\overline{d} \subseteq M$ we also have that $\overline{d}^* \subseteq M'$. Also since f is an isomorphism $tp_{N^*}(\overline{d}^*/M_1) = tp_{N'}(\overline{d}^*/M_1) = tp_N(\overline{d}/M_1) = p$. We also have that $N^* \models d_p\overline{x}R(\overline{x}, \overline{e})$ and hence that $N^* \models R(\overline{d}^*, \overline{e})$. But since R is absolute we have that $M^* \models R(\overline{d}^*\overline{e})$. Also since T is trivial of Morley Rank 1 we have that $M' \downarrow_{M_0} M_1$ in M^*.

Let $r(\overline{x}) = tp_{M^*}(\overline{d}^*/M_1)$. Thus $M^* \models d_r\overline{x}R(\overline{x}, \overline{e})$ and hence $M \models d_r\overline{x}R(\overline{x}, \overline{e})$. Also $M \models \sigma(\overline{c})$ and so $M \models \sigma \restriction_{\overline{x}} (\overline{d})$. Since $tp_M(\overline{d}/M_1) = tp_{M'}(\overline{d}^*/M_1)$ we get that $M' \models \sigma \restriction_{\overline{x}} (\overline{d}^*)$ and hence that $\sigma \restriction_{\overline{x}} \in r$. Thus we have that $M^* \models d_r\overline{x}[R(\overline{x}, \overline{e}) \wedge \sigma(\overline{x}, \overline{e})]$. But M is elementary in M^* so the same holds of M and hence $M \models d_{r|M_0}\overline{x}[R(\overline{x}, \overline{e}) \wedge \sigma(\overline{x}, \overline{e})]$. But this contradicts that $M \models \overline{c}$ is (R, σ) mutually algebraic.

To finish suppose that $\overline{z} = \overline{x}^\frown\overline{y}$ is a partition of $\overline{z}$ (so $\overline{c} = \overline{d}^\frown\overline{e}$). Assume for contradiction that $N \models d_p\overline{x}[R^*(\overline{x}, \overline{e}) \wedge \sigma^*(\overline{x}, \overline{e})]$ for some $p \in S_{|\overline{x}|}(M_0)$. Choose $N' \succeq N$ with $\overline{a} \in N'$ such that $\overline{a} \models p|N$. In particular we have that $N' \models R^*(\overline{a}, \overline{e})$ and so we have $N' \models \neg R(\overline{a}, \overline{e_i})$ for all $i < n$ while also $N' \models R(\overline{a}, \overline{e})$. But note that since $lg(\overline{a}) < n$ and T has Morley Rank 1, $tp(\overline{a}/M_1)$ has weight less than n. So for some $i < n$ we must have that $\overline{a} \downarrow_{M_1} \overline{e}_i$ in N'. Thus for this i we have that $tp_{N'}(\overline{ae}/M_1) = tp_{N'}(\overline{ae}_i/M_1)$ But R is a formula over M_1 so we get a contradiction. □

Remark 3. Notice that in the above Lemma there is nothing special about the formula σ. In particular if we let $\phi(x)$ be any $\mathcal{L}(M_0)$-formula and for $I \subseteq n$ set

$$\phi_I(\overline{x}) := \bigwedge_{i \in I} \phi(x_i) \wedge \bigwedge_{i \notin I} \neg\phi(x_i).$$

The above Proof yields: Suppose $M, N \models T(M_0), M \subseteq N, R$ is an absolute formula and ϕ is any $\mathcal{L}(M_0)$-formula. If $M \models \overline{c}$ is $R \wedge \phi_I$ mutually algebraic for some $I \subseteq n$ (where $n = lg(\overline{c})$) then there is an absolute $\mathcal{L}(M_0)$ formula R^* and $I^* \subseteq n$ such that $N \models \overline{c}$ is $R \wedge \phi_{I^*}$ mutually algebraic. In particular (setting $\phi(x)$ to be $x = x$ in the preceding statement) we get that if R is absolute and $M \models \overline{c}$ is R mutually algebraic then for some R^*, $N \models \overline{c}$ is R^* mutually algebraic.

Lemma 6. *Suppose $M \subseteq N$ are models of $T(M_0)$, $R(\overline{z})$ is an absolute formula, and $\overline{c} \in M^n$ such that $M \models \sigma(\overline{c})$ and $N \models \sigma(\overline{c})$. Then $M \models \overline{c}$ is (R, σ) mutually algebraic if and only if $N \models \overline{c}$ is (R, σ) is mutually algebraic.*

Proof: This follows immediately from the fact that any formula which does not fork over M_0 is satisfiable in M_0. □

With these Lemmas we may establish the main result of this section.

Proposition 1. *Suppose that T is uncountably categorical. Furthermore suppose that the theory $\hat{T}$ obtained from our theory $T(M_0)$ by adding predicates for A and B is model complete, then the definable sets $A(x)$ and $B(x)$ are absolute between M and N.*

Proof: To begin we establish the following:

Claim 1. *(Existence) Let $a \in M \setminus M_0$ such that $M \models A(a)$ also let $p \in S^1(M_0)$ be a non-algebraic type such that $p(x) \vdash B(x)$. Then there is a quantifier free $\mathcal{L}(M_0)$ formula $R(\overline{z})$ and $\sigma \in 2^n$ and $\overline{c} \in (M \setminus M_0)^n$ such that:*

1. *$c_0 = a$.*
2. *$M \models \overline{c}$ is (R, σ) mutually algebraic.*
3. *$c_{n-1} \models p$.*

Proof of Claim 1: By the uncountable categoricity of T we have that $tp_M(a/M_0) \not\perp p$. So since T is of rank 1 there is an $\mathcal{L}(M_0)$-formula $\theta(u, v)$ such that $\theta(a, b)$ for some b realizing p and such that $M \models \exists^{=k} u \theta(u, b)$. We may choose θ so that k is as small as possible and also without loss of generality we may assume that

$$\forall u \forall v (\theta(u, v) \rightarrow D_0(u) \wedge D_1(v)).$$

where D_0 and D_1 are $\mathcal{L}(M_0)$ definable strongly minimal sets for which a, b are generic respectively. We may also assume that

$$\forall v(\exists u\theta(u,v) \to \exists^{=k} u\theta(u,v)).$$

Note since $\hat{T}$ is model complete we can find a quantifier free $\mathcal{L}(M_0)$ formula $R(u\bar{z}v)$ and $\sigma \in 2^{n+2}$ (where $n = lg(\bar{z})$) such that

$$\hat{T} \vdash \forall u \forall v[\theta(u,v) \leftrightarrow \exists \bar{z}(R(u\bar{z}v) \wedge \sigma(u\bar{z}v))].$$

Choose R such that $lg(\bar{z}) = n$ is minimal.

Sub-Claim. If $a', b' \in M \setminus M_0$ and $M \models R(a'\bar{c}b') \wedge \sigma(a'\bar{c}b')$ then $\bar{c} \cap M_0 = \emptyset$.

Suppose not and without loss of generality assume that $c_0 \in M_0$. Consider the formula $\psi(u,v) = \exists z_1 ... \exists z_{n-1}(R(uc_0z_1...z_{n-1}v) \wedge \sigma(uc_0z_1...z_{n-1}v))$. By assumption $\hat{T} \vdash \forall u \forall v(\psi(u,v) \to \theta(u,v))$. For any b realizing p we can find a'' such that $tp(a''/M_0) = tp(a/M_0)$ and $M \models \psi(a,b)$. We also have that if b realizes p then $\psi(M,b) \subseteq \theta(M,b)$. From these two facts and the minimality of k we must have that for b realizing p that $\theta(M,b) = \psi(M,b)$. Since the set $D_1(M)$ is strongly minimal and p is its generic type there are only finitely many elements $m_1...m_l \in D_1(M_0)$ such that $\psi(M,m_i) \neq \theta(M,m_i)$. For $1 \leq i \leq l$ let $h_1...h_{j_i}$ list $\theta(M,m_i)$, note that for each i this must be finite and a subset of M_0. Now consider the following formula

$$\widetilde{\psi}(u,v) = \exists z_1...z_{n-1}([(\bigwedge_{i=1}^{l} v \neq m_i) \to R(uc_0z_1...z_{n-1}v)] \wedge$$

$$[\bigwedge_{i=1}^{l}(v = m_i \to \bigvee_{r=1}^{j_i} u = h_r)] \wedge \sigma(uc_0z_1...z_{n-1}v)).$$

(If it is the case that for some m_i the set $\theta(M,m_i)$ is empty we replace the clause $\bigvee_{r=1}^{j_i} u = h_r$ in the above formula by $u \neq u$.)

Under our assumption we must have that $M \models \forall u \forall v(\theta(u,v) \leftrightarrow \widetilde{\psi}(u,v))$, but this violates the minimality of n, a contradiction which establishes the sub-claim.

Sub-Claim. If $a', b' \in M \setminus M_0$ and $M \models R(a'\bar{c}b') \wedge \sigma(a'\bar{c}b')$ then $a'\bar{c} \subseteq acl(M_0b') \setminus M_0$.

Suppose the sub-claim is false. The fact that $a'\bar{c} \cap M_0 = \emptyset$ is immediate from the previous claim. By choice of θ we must have that $a' \in acl(M_0b')$, so for the claim to be false we must have that some $c_i \in \bar{c}$ is such that $c_i \notin acl(M_0b')$. Fix such a c_i. Let $X = \{a', c_1...c_n\} \cap acl(M_0c_i)$. By assumption and the above claim no element of X is in M_0. So by the exchange property and the triviality of T we must have that $a' \notin X$. For notation set $\bar{c} = \bar{d}^\frown\bar{e}$, where $\bar{d}$ enumerates X, and $\bar{e}$ enumerates X's complement in $\bar{c}$.

We get that $\bar{d} \downarrow_{M_0} a'b'\bar{e}$ in M. Choose a countable $M_1 \preceq M_0$ such that $\bar{d} \downarrow_{M_1} M_0a'b'\bar{e}$ in M, M_1 contains any parameters from M_0 appearing in R or σ.

By the saturation of M_0 there is $\overline{d}^* \subseteq M_0$ such that $tp(\overline{d}^*/M_1) = tp(\overline{d}/M_1)$. Hence we get that $tp(\overline{d}^* a'b'\overline{e}/M_1) = tp(\overline{d}a'b'\overline{e}/M_1)$. But then we must have that $M \models \theta(a'\overline{d}^*\overline{e}b') \wedge \sigma(a'\overline{d}^*\overline{e}b')$, which violates the previous claim. Hence we have a contradiction and our sub-claim is proved.

To establish the Claim, given a we may find b realizing p such that $M \models \theta(a, b)$. Next choose $\overline{d}$ such that $M \models R(a\overline{d}b) \wedge \sigma(a\overline{d}b)$. Then setting $\overline{c} = a\overline{d}b$ we obtain the desired result for Claim 1. $\square$

We may now finish the proof of Proposition 1.

We fix $a \in M$ such that $M \models A(a)$. Our goal is to show that $N \models A(a)$. If $a \in M_0$ the result is immediate so we may assume that $a \notin M_0$. Let $p_1, ..., p_m$ list all the non-algebraic elements of $S^1(M_0)$ such that $p_i(x) \vdash B(x)$. Now apply Claim 1 to a and p_i for each i. So we get $R^i(\overline{z}), \sigma^i$, and $\overline{c}^i$ for $1 \leq i \leq m$ such that $M \models \overline{c}^i$ is (R^i, σ^i) mutually algebraic and $c^i_0 = a$. By repeatedly applying Remark 1 we may assume that $lg(\overline{c}_i) = lg(\overline{c}_j) = n$ for all i, j. Let $b^i = c^i_{n-1}$, so $b^i \models p_i$. By Lemma 4 we have that $\overline{c}_i \subseteq M \setminus M_0$.

Claim. There is a finite set $\mathcal{F}$ of pairs (R, σ) where R is quantifier free and $\sigma \in 2^n$ such that if $\overline{c} \in M^n$ and for some i, $M \models \overline{c}$ is (R, σ) mutually algebraic with $(R, \sigma) \in \mathcal{F}$ and $c_{n-1} = b^i$ then $N \models \overline{c}$ is (R^*, σ^*) mutually algebraic for some $(R^*, \sigma^*) \in \mathcal{F}$.

To establish the claim we inductively define an ascending sequence of finite sets $\mathcal{F}_k$ for $k \geq -1$ with the following properties:

(1) $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$.
(2) $\mathcal{F}_{-1} = \emptyset$.
(3) $\mathcal{F}_0 = \{(R^i, \sigma^i) : 1 \leq i \leq m\}$.
(4) If $\overline{c} \subset M$ is such that $M \models \overline{c}$ is (R, σ) mutually algebraic for some $(R, \sigma) \in \mathcal{F}_k$ and $c_{n-1} = b^i$ for some i then $N \models \overline{c}$ is (R^*, σ^*) mutually algebraic for some $(R^*, \sigma^*) \in \mathcal{F}_{k+1}$.
(5) If $(R^*, \sigma^*) \in \mathcal{F}_{k+1} \setminus \mathcal{F}_k$ with $k \geq 0$ then $\sigma < \sigma^*$ for some σ such that $(R, \sigma) \in \mathcal{F}_k \setminus \mathcal{F}_{k-1}$ for some R.

First we show that if we can construct the $\mathcal{F}_k$ as above we have the claim. But this is trivial since condition (5) above guarantees that there must be a k^* such that if $k \geq k^*$ then $\mathcal{F}_k = \mathcal{F}_{k^*}$, since otherwise we would have to have arbitrarily long increasing $\leq$ chains of elements of 2^n. Hence we can set $\mathcal{F} = \mathcal{F}_{k^*}$. To construct the $\mathcal{F}_k$ we let $\mathcal{F}_{-1}$ and $\mathcal{F}_0$ be as required. Next given $\mathcal{F}_k$ for $k \geq 1$ we show how to obtain $\mathcal{F}_{k+1}$. We tentatively define $\mathcal{F}_{k+1} = \mathcal{F}_k$ and determine if we need to add more elements to satisfy condition (4). Let $\overline{c}$ be such that $M \models \overline{c}$ is (R, σ) mutually algebraic and $c_{n-1} = b^i$ for some $(R, \sigma) \in \mathcal{F}_k$ (note there are only finitely many such $\overline{c}$'s). If $(R, \sigma) \in \mathcal{F}_{k-1}$ then condition (4) is satisfied by the inductive hypothesis, so we may assume that $(R, \sigma) \in \mathcal{F}_k \setminus \mathcal{F}_{k-1}$. If $N \models \sigma(\overline{c})$ then by Lemma 6 we have that $N \models \overline{c}$ is (R, σ) mutually algebraic and hence clause (4) above holds for this

$\overline{c}$ without adding anything new to $\mathcal{F}_{k+1}$. If on the other hand $N \models \sigma^*(\overline{c})$ for some $\sigma^* \neq \sigma$ then first of all notice that $\sigma < \sigma^*$ and by Lemma 5 we can find R^* such that $N \models \overline{c}$ is (R^*, σ^*) mutually algebraic. So add (R^*, σ^*) to $\mathcal{F}_{k+1}$. Note that this construction then satisfies the conditions (1)-(5) above. Hence we have the claim.

Let $G^i(M)$ be the following set:

$$\{\overline{c} \in M^n : \text{ for some } (R, \sigma) \in \mathcal{F}, M \models \overline{c} \text{ is } (R, \sigma) \text{ mutually algebraic and } c_{n-1} = b^i\}.$$

Note that by Lemma 4 $G^i(M)$ is finite. Furthermore note that by Lemma 3 for any i, $G^i(M)$ is the set:

$$\bigcup_{(R,\sigma)\in\mathcal{F}} \{\overline{c} : M \models \Gamma_{R,\sigma}(\overline{c}b^i)\}.$$

In particular the cardinality of $G^i(M)$ is property of the type of b^i over M_0. Since each $G^i(M)$ is finite after renumbering we may assume that for some $m_0 \leq m$ we have that $G^i(M)$ is of largest possible size if and only if $1 \leq i \leq m_0$.

Claim. For $1 \leq i \leq m_0$ we have that $G^i(M) = G^i(N)$.

For the claim note that by the properties of $\mathcal{F}$ we already have that $G^i(M) \subseteq G^i(N)$. So if $G^i(M) \neq G^i(N)$ we must have that $G^i(N)$ is of strictly larger size than $G^i(M)$. Since $M \models Bb_i$ we have that $N \models Bb_i$ and hence that $tp_N(b_i/M_0) = p_j$ for some $1 \leq j \leq m$. But $G^i(N) = \bigcup_{(R,\sigma)\in\mathcal{F}}\{\overline{c} : N \models \Gamma_{R,\sigma}(\overline{c}b^i)\}$, thus for any $b \in M$ realizing p_j we must have that $\left|\bigcup_{(R,\sigma)\in\mathcal{F}}\{\overline{c} : M \models \Gamma_{R,\sigma}(\overline{c}b)\}\right| = |G^i(N)|$. But then in particular we would have that $|G^j(M)| > |G^i(M)|$, contradicting the maximality of the size of $G^i(M)$.

As a consequence note we must that if $1 \leq i \leq m_0$ there is $1 \leq j \leq m_0$ such that $tp_N(b^i/M_0) = tp_M(b^j/M_0)$.

For notation let:

$$\mathcal{F}^* = \{\sigma \in 2^n : \text{ there is } R \text{ such that } (R, \sigma) \in \mathcal{F}\}.$$

Also for each $\sigma \in \mathcal{F}^*$ let $G^i_\sigma(M)$ be the set:

$$\{\overline{c} \in G^i(M) : M \models \overline{c} \text{ is } (R, \sigma) \text{ mutually algebraic for some } R \text{ such that } (R, \sigma) \in \mathcal{F}\}.$$

Note that $G^i_\sigma(M) = \bigcup_{(R,\sigma)\in\mathcal{F}}\{\overline{c} : M \models \Gamma_{R,\sigma}(\overline{c}b^i)\}$.

List $\mathcal{F}^*$ as $\sigma_1 ... \sigma_l$ such that if $\sigma_r \leq \sigma_s$ then $r \leq s$. After renumbering we may assume that for some $m_1 \leq m_0$ we have that $G^i_{\sigma_l}$ is of maximal size for $1 \leq i \leq m_0$ if and only if $i \leq m_1$.

Claim. For $1 \leq i \leq m_1$ we have that $G^i_{\sigma_l}(M) = G^i_{\sigma_l}(N)$.

For the claim first note that $G^i_{\sigma_l}(M) \subseteq G^i_{\sigma_l}(N)$ since if $\overline{c} \in G^i_{\sigma_l}(M)$ then $\overline{c} \in G^i(N)$ and hence in G^i_τ for some τ but $\tau \geq \sigma_l$ by Remark 2 and hence we must have $\tau = \sigma_l$. So if $G^i_{\sigma_l}(M) \neq G^i_{\sigma_l}(N)$ then we must have that $\left|G^i_{\sigma_l}(M)\right| < \left|G^i_{\sigma_l}(N)\right|$. Now argue exactly as above to conclude that for some $1 \leq j \leq m_0$ we must have that $\left|G^j_{\sigma_l}(M)\right| = \left|G^i_{\sigma_l}(N)\right| > \left|G^i_{\sigma_l}(M)\right|$, a contradiction.

Now we can renumber again to find $m_2 \leq m_1$ such that $G^i_{\sigma_{l-1}}$ is of maximal size for $1 \leq i \leq m_1$ if and only if $i \leq m_2$. Similarly we can now also show that for $1 \leq i \leq m_2$ we have that $G^i_{\sigma_{l-1}}(M) = G^i_{\sigma_{l-1}}(N)$.

We can repeat this argument to find $m^* \leq m$ such that if $1 \leq i \leq m^*$ then $G^i_\sigma(M) = G^i_\sigma(N)$ for all $\sigma \in \mathcal{F}^*$.
Fix $1 \leq i \leq m^*$. We have that $\overline{c}^i \in G^i_{\sigma^i}(M)$ and hence that $\overline{c}^i \in G^i_{\sigma^i}(N)$. In particular we have that $N \models A(a)$. □

4. Proof of the Main Theorem

We now have established all of the necessary tools we need in order to prove the main theorem. At this point the proof closely follows that in [2].
In this section T is a trivial uncountably categorical theory of Morley Rank 1. We fix M_0 to be an uncountable (and hence saturated) model of T and aim to show that $T(M_0)$ is model complete by showing that if N and M are models of $T(M_0)$ of cardinality $\kappa > |M_0|$ with $M \subseteq N$ then $M \preceq N$. We let $\mathcal{L}^*$ be the language $\mathcal{L}$ of T expanded by constants for M_0.

Note that as pointed out in [2] in order to show that for any model M_0 of T that $T(M_0)$ is model complete it suffices to show the result some specific model M_0, hence our assumption that M_0 is uncountable does not limit the generality of our result.

We repeat the following definitions from [2] (which we include for simplicity of exposition):

Definition 6. An $\mathcal{L}^*$ formula $\phi(\overline{x}, \overline{y})$ is an *(n,m)-formula* if $lg(\overline{x}) = n$ and $lg(\overline{y}) = m$. We identify the following families of statements:

- $A_{n,m}$, the statement that for all absolute (n,m)-formulas $\phi(\overline{x}, \overline{y})$, the formula $\exists^{<\infty}\overline{y}\phi(\overline{x}, \overline{y})$ is absolute.
- $B_{n,m}$, the statement that for all absolute (n,m)-formulas $\phi(\overline{x}, \overline{y})$ if $\overline{b} \in M^n$ and $N \models \exists^{<\infty}\overline{y}\phi(\overline{b}, \overline{y})$ then $\phi(\overline{b}, N) = \phi(\overline{b}, M)$.
- $C_{n,m}$, the statement that for all absolute (n,m)-formulas $\phi(\overline{x}, \overline{y})$, the formula $\exists\overline{y}\phi(\overline{x}, \overline{y})$ is absolute.

We plan to show that same sequence of Lemmas interrelating these various statements as in [2], and hence deduce model completeness. We begin with a simple special case.

Lemma 7. *Suppose that the Morley Degree of T is m and that there are absolute $\mathcal{L}^*$-formulas $\phi_1(x)...\phi_m(x)$ such that for all i, $M \models \exists^\infty x\phi_i(x)$ and for all $i \neq j$, $M \models \neg\exists x(\phi_i(x) \wedge \phi_j(x))$. Then $M \preceq N$.*

Proof: In this special case we simply repeat the proof for the strongly minimal case in [2]. □

We now proceed to reprove the main lemmas in [2] to obtain our desired result. We consider first the following two lemmas.

Lemma 8. *For all $n, m \in \omega$, $B_{n,m}$ implies $C_{n,m}$.*

Lemma 9. *For all $n, m \in \omega$, $B_{n,m}$ implies $A_{n,m+1}$.*

The proof of both these lemmas mirror the analogous lemmas in [2] with only minimal changes.
The proof of the following Lemma also closely mirrors that of its analogue in [2] but we include an outline pointing out the relevant changes.

Lemma 10. *For all $n, m, B_{n,m+1}$ and $A_{n+1,m}$ imply $B_{n+1,m}$.*

Proof: As in [2] we may assume that $n \geq 1$. Choose $\phi(\overline{x}, y, \overline{z}), \overline{b}, b_2, \overline{c}^*$ as in the strongly minimal case. Choose $\overline{e}_1 ... \overline{e}_n \in (M \setminus acl(M_0\overline{b}))^n$ independent realizing all the generic types $p \in S^1(M_0)$. (We can do this due to cardinality assumptions and the uncountable categoricity.)

We first deal with an easy case:

Case 1 $N \models \exists^\infty \overline{z} \phi(\overline{b}, e_i, \overline{z})$ for all i.

In this case we can proceed exactly as in the analogous case in [2].
Case 2 For some i, $N \models \exists^{<\infty} \overline{z} \phi(\overline{b}, e_i, \overline{z})$.

Fix e_i such that we have $N \models \exists^{<\infty} \overline{z} \phi(\overline{b}, e_i, \overline{z})$.
Working with this e_i in place of the e^* in the strongly minimal case we can follow the proof in [2] to get formulas δ^i_j for $1 \leq j \leq r(i)$ (the i superscript indicates the dependence on i).
Note also that if $N \models \delta^i_j(\overline{b}, b_2, \overline{c}^*)$ for some j then as in the strongly minimal case we are done.
Now fix $m \leq n$ such that for $i \leq m$ we have that $N \models \exists^{<\infty} \overline{z} \phi(\overline{b}, e_i, \overline{z})$ and if $i > m$ $N \models \exists^\infty \overline{z} \phi(\overline{b}, e_i, \overline{z})$.

So we may assume that $N \models \neg \delta^i_j(\overline{b}, b_2, \overline{c}^*)$ for all $i \leq m$ and all j.
Now let $\eta(\overline{x}, y, \overline{z})$ be the formula:

$$\phi(\overline{x}, y, \overline{z}) \wedge \exists^{<\infty} \overline{z} \phi(\overline{x}, y, \overline{z}) \wedge \bigwedge_{i \leq m} \bigwedge_{j < r(i)} \neg \delta^i_j(\overline{x}, y, \overline{z}_j).$$

Then $N \models \eta(\overline{b}, b_2, \overline{c}^*)$ and η is absolute as in the strongly minimal case.

We claim: $N \models \exists^{<\infty} y \overline{z} \eta(\overline{b}, y, \overline{z})$. Set

$$F = \{f \in N : N \models \exists \overline{z} \eta(\overline{b}, f, \overline{z})\}.$$

As in [2] we show that F is finite. For convenience let D_i for $1 \leq i \leq n$ be strongly minimal pairwise disjoint sets such that e_i is generic for D_i. We must show that $F \cap D_i$ is finite for each i. For $i \leq m$ the finiteness of $F \cap D_i$ is exactly as in the strongly minimal case (replace e^* with e_i). So now suppose that $i > m$. As in [2] we would have to get that $N \models \exists \overline{z} \eta(\overline{b}, e_i, \overline{z})$. But in particular this means that $N \models \exists^{<\infty} \overline{z} \phi(\overline{b}, e_i, \overline{z})$ which contradicts that $i > m$.

So F is finite. Now we can finish exactly as in the strongly minimal case. $\square$

All that is left to prove is the following Lemma, its proof requires Proposition 1 from the previous section.

Lemma 11. *For all $m \in \omega$ $B_{1,m}$ holds.*

Proof: Notice that exactly as in [2], if for any theory T we can establish this lemma then in conjunction with the previous Lemmas we may deduce that $T(M_0)$ is model complete.
Let n be the Morley Degree of T. We also need to define a "quantifier free degree" for the theory. Namely for any trivial, uncountably categorical theory T' of Morley Rank 1 let $n_{qf}(T')$ be the maximal number of pairwise disjoint infinite quantifier free $\mathcal{L}(M')$-definable sets in a model of $T(M')$, where M' is some model of T'.
Our proof will be by induction on $n - n_{qf}(T)$.

The case $n - n_{qf}(T) = 0$ is handled by Lemma 7.

For induction suppose we know the result (and hence also the model completeness result) for $n - n_{qf}(T) \leq m$ and we are now in a situation where $n - n_{qf}(T) = m + 1$. Fix an absolute $\mathcal{L}(M_0)$-formula $\phi(x, \overline{y})$. Choose $r \in \mathbb{N}$ such that $M \models \exists^{\leq r} \overline{y} \phi(\overline{y}, a)$. Let $A(x)$ be the set defined by $\exists^{\leq r} \overline{y} \phi(\overline{y}, x)$ and $B(x)$ its complement. Suppose that $M \models A(a)$ for some $a \in M$, to establish the Lemma it suffices to show show that $N \models A(a)$. The case where $A(M)$ is finite or co-finite is immediate, so without loss of generality we may assume that $A(M)$ is infinite and co-infinite. If we can find quantifier free $\mathcal{L}(M_0)$-definable subsets $Y_1 ... Y_l$ of M such that the symmetric difference of $A(M)$ and $Y_1 \cup ... \cup Y_r$ is finite then we are done, hence we may assume that this is never the case. Let $\hat{T}$ be the theory obtained from $T(M_0)$ by adding predicates for A and B.

Claim. $n_{qf}(T) < n_{qf}(\hat{T})$.

Proof: Let $X_1 ... X_{n_{qf}(T)}$ be pairwise disjoint infinite quantifier free $\mathcal{L}(M_0)$-definable subsets of M. Our claim is immediate if for for some $1 \leq i \leq n_{qf}(T)$ we have that $X_i \cap A$ and $X_i \setminus A$ is infinite. So we may assume that this is never the case, i.e. for all $1 \leq i \leq n_{qf}(T)$ either $X_i \cap A$ is finite or $X_i \setminus A$ is finite. After renumbering we may assume that there is $1 \leq l \leq n_{qf}(T) + 1$ such that $X_i \cap A$ is finite if and only if $i < l$. Our Claim is also immediate if $A \setminus (X_{l+1} \cup ... \cup X_{n_{qf}(T)})$ is infinite. But if this is not the case then the symmetric difference of A and $X_{l+1} \cup ... \cup X_{n_{qf}(T)}$ is finite, contradicting one of our initial assumptions.

By induction we have that $\hat{T}$ is model complete. But then by Proposition 1 we get our desired result. $\square$

Proof of Theorem 1: As noted above the main theorem follows immediately from the preceding Lemmas exactly as in [2]. □

Exactly as in the strongly minimal case our Theorem has the following corollary.

Corollary 3. *If T is trivial, uncountably categorical, and the Morley Rank of T is* 1 *then:*

(1) *T is Σ_3 axiomatizable.*
(2) *If $\mathcal{L}$ is a computable language and T has a computable model M then T is decidable in $\emptyset''$.*

Finally notice that Theorem 1 does not generalize to any trivial uncountably categorical theory, even if we assume that theory is also ω-categorical. This fact is demonstrated by examples constructed by Marker in [3] which are trivial, totally categorical, and not Σ_3 axiomatizable.

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