MATH 141H SOLUTIONS TO SAMPLE EXAM

1. (a) converges for $r < 1$ (Do the integral explicitly)
(b) converges for $r > 1$ (Do the integral explicitly)

2. Take logarithm, then exponentiate:

$$\lim \frac{\ln(n/(n+1))}{1/(n-1)} = \frac{0}{0}$$

so l'Hospital applies. The numerator is $\ln n - \ln(n+1)$, and its derivative is $\frac{1}{n} - \frac{1}{n+1}$. The derivative of the denominator is $-1/(n-1)^2$. Some algebra shows that we now have $\lim \frac{-(n-1)^2}{n(n+1)} = -1$. Exponentiating yields the answer e^{-1}.

3. It's probably easiest to use the shell method. Use $f(x) = \sqrt{R^2 - x^2}$. The part of the graph with $-3 \le y \le 3$ is revolved around the y-axis. When $f(x) = y = 3$ we have $x = \sqrt{R^2 - 3^2}$. The largest x can be is R. Therefore, the volume aove the x-axis is

$$V = 2\pi \int_{\sqrt{R^2-9}}^{R} x\sqrt{R^2 - x^2}\, dx = \pi \, \frac{-2}{3}(R^2 - x^2)^{3/2}\Big|_{\sqrt{R^2-x^2}}^{R} = 18\pi$$

Double to include the lower half. The final answer is 36π.

4. Partial fractions yields

$$\frac{2x^2+3}{x(x-1)^2} = \frac{3}{x} + \frac{-1}{x-1} + \frac{5}{(x-1)^2}.$$

Integrating yields

$$3\ln|x| - \ln|x-1| - 5(x-1)^{-1} + C.$$

5. Partial fractions, with a lot of calculation, yields

$$\frac{2x}{(x+1)(x^2+1)^2} = \frac{-1/2}{x+1} + \frac{1}{2}\frac{x-1}{x^2+1} + \frac{x+1}{(x^2+1)^2}.$$

Integrating these yields

$$-\frac{1}{2}\ln|x+1| + \frac{1}{4}\ln|x^2+1| + \frac{1}{2}\frac{x-1}{x^2+1} + C$$

6. $S = 2\pi \int_0^2 f(x)\sqrt{1+f'(x)^2}\,dx = 2\pi \int_0^2 x^3\sqrt{1+9x^4}\,dx$
Use the substitution $u = 9x^4$. The final answer is $(\pi/27)(145^{3/2}-1)$.

7. $V = \int_0^{30} A(x)\,dx = \int_0^{30}(30-x)^2\,dx = 9000$.

8. The ball goes down 6, then up $(3/4)6$ and down $(3/4)6$, then up $(3/4)^2 6$ and down the same amount, etc. The total distance is

$$6 + 2\left(\frac{3}{4}6 + (\frac{3}{4})^2 6 + \cdots\right) = 6 + 12(\frac{3/4}{1-(3/4)}) = 42.$$

(the infinite sum is a geometric series)

9. (a) Converges: Limit comparison test with $\sum 1/n^{3/2}$.
(b) Diverges. The terms are larger than the terms of $\sum 1/n$, which diverges. The comparison test implies the answer.
(c) Converges: use the integral test:

$$\int_2^\infty \frac{dx}{x(\ln x)^2} = -\left.\frac{1}{\ln x}\right|_2^\infty = \frac{1}{\ln 2} < \infty,$$

so the integral converges. Therefore, the sum converges.

10. Draw the boxes under the curve $y = 1/x^2$, starting at $x = 1000$. The sum is less than

$$\int_{1000}^\infty \frac{dx}{x^2} = .001$$