Modular Forms Homework 2

1. Let p be prime and let x/y be a rational number, where x, y are relatively prime integers.
   (a) Show that if $p \nmid y$ then there is an element of $M \in \Gamma_0(p)$ such that $M : 0 \mapsto x/y$.
   (b) Show that if $p \mid y$ then there is an element of $M \in \Gamma_0(p)$ such that $M : i\infty \mapsto x/y$.
   (c) Show that $\Gamma_0(p)$ has exactly two cusps.
2. Let $M_k(\mathbf{Z})$ be the set of modular forms of weight k for $SL_2(\mathbf{Z})$ whose Fourier coefficients are integers. Show that $M_k(\mathbf{Z})$ is a free $\mathbf{Z}$-module of rank equal to the $\mathbf{C}$-dimension of M_k. (*Hint:* E_4, E_6, and Δ have integral Fourier coefficients.)
3. Let $E_6 = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n$ and $E_{12} = 1 + \frac{65520}{691} \sum_{n \geq 1} \sigma_{11}(n) q^n$ be the Eisenstein series of weight 6 and 12 for $SL_2(\mathbf{Z})$.
   (a) Show that
   $$E_6^2 = E_{12} - \frac{a}{691}\Delta$$
   for an integer $a \equiv 65520 \pmod{691}$.
   (b) Show that $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$ for all $n \geq 1$, where $\Delta = \sum_{n \geq 1} \tau(n) q^n$.

Due: October 20, 2009