Math 246, First In-Class Exam Solutions (Spring 2003)
Professor Levermore

(1) (12 points) Consider the differential equation

$$\frac{dy}{dt} = -y^2(y-2)(y-4) \,.$$

(a) Find all of the equilibrium (stationary) solutions and classify each as stable, unstable, or semistable.

Solution: It is clear from the right-hand side that $y = 0$, $y = 2$, and $y = 4$ are equilibrium solutions. A sign analysis then shows that:

- $\frac{dy}{dt} = -y^2(y-2)(y-4) < 0$ when y is in $(-\infty, 0)$, $(0, 2)$, and $(4, \infty)$;
- $\frac{dy}{dt} = -y^2(y-2)(y-4) > 0$ when y is in $(2, 4)$;

Hence $y = 0$ is a semistable equilibrium solution, $y = 2$ is an unstable equilibrium solution, while $y = 4$ is a stable equilibrium solution.

(b) Draw a graph of y versus t showing the direction field and several solution curves, including all of the equilibrium solutions and solutions above and below each equilibrium value.

Solution: Will be given in class.

(c) If $y(0) = 3$, what is the limiting value of y as $t \to \infty$?

Solution: It is clear from the answer to (a) that

- $\frac{dy}{dt} > 0$ when y is in $(2, 4)$.

Hence, $y \to 4$ as $t \to \infty$ when $y(0) = 3$.

(2) (18 points) Give an explicit solution to each of the following initial value problems. Identify the interval over which each solution is defined.

(a) $\frac{du}{dz} = \frac{\cos(z)}{1+u}$, $\quad u(0) = 0$.

Solution: The equation is *separable*, so write it in the separated differential form as

$$(1+u)\,du = \cos(z)\,dz \,.$$

This equation can be integrated to obtain

$$u + \tfrac{1}{2}u^2 = \sin(z) + C \,.$$

The value of the integration constant C is found through the initial condition $u(0) = 0$ by setting $z = 0$ and $u = 0$, whereby

$$C = 0 + \tfrac{1}{2}0^2 - \sin(0) = 0 \,.$$

Hence, the solution is given implicitly by

$$u + \tfrac{1}{2}u^2 = \sin(z) \,.$$

This may be solved explicitly for u by completing the square:

$$\tfrac{1}{2}(1+u)^2 = \sin(z) + \tfrac{1}{2} \,;$$

$$(1+u)^2 = 1 + 2\sin(z) \,;$$

$$u = -1 + \sqrt{1 + 2\sin(z)} \,.$$

Notice that the positive square root is taken in order to satisfy the initial condition $u(0) = 0$. The interval over which this solution is defined must contain $z = 0$ and satisfy

$$1 + 2\sin(z) > 0\,.$$

This is the interval $-\frac{\pi}{6} < z < \frac{7\pi}{6}$.

(b) $\dfrac{dz}{dt} = \dfrac{e^t - z}{2+t}\,, \quad z(0) = 3\,.$

Solution: This equation is *linear* in z, so write it in the normal form

$$\frac{dz}{dt} + \frac{z}{2+t} = \frac{e^t}{2+t}\,.$$

One sees that an integrating factor μ is given by

$$\mu = \exp\left(\int \frac{1}{2+t}\,dt\right) = \exp\big(\ln(2+t)\big) = 2+t\,.$$

Upon multiplying the equation by $\mu = (2+t)$, one finds that

$$\frac{d}{dt}\big((2+t)z\big) = e^t\,,$$

which is then integrated to obtain

$$(2+t)z = e^t + C\,.$$

The value of the integration constant C is found through the initial condition $z(0) = 3$ by setting $t = 0$ and $z = 3$, whereby

$$C = (2+0)3 - e^0 = 6 - 1 = 5\,.$$

Hence, upon solving explicitly for z, the solution is

$$z = \frac{e^t + 5}{2+t}\,.$$

The interval over which this solution is defined must contain $t = 0$ and satisfy $t \neq -2$. This is the interval $t > -2$.

(3) (18 points) Suppose a room containing 50 m^3 (m = meters) of air is initially free of carbon monoxide. Beginning at $t = 0$ cigarette smoke containing 4% carbon monoxide is introduced into the room at a rate of .001 m^3/min, and the well-circulated mixture is allowed to leave the room at the same rate. Let $C(t)$ denote the volume of carbon monoxide in the room at time $t \geq 0$.

(a) Is $C(t)$ an increasing or decreasing function of t? (Give reasoning.)

Solution: The carbon monoxide concentration in the room at time t is $C(t)/50$. Because this is also the concentration of the outflow, $C(t)$, the volume of carbon monoxide in the room, will satisfy

$$\frac{dC}{dt} = \tfrac{1}{1000}\cdot\tfrac{4}{100} - \tfrac{1}{1000}\cdot\frac{C}{50} = \tfrac{1}{50000}(2 - C)\,.$$

From this one sees that

$$\frac{dC}{dt} > 0 \quad \text{for } C < 2,$$

whereby $C(t)$ is an increasing function of t that will approach the equilibrium value of 2 m^3 as $t \to \infty$.

An alternative physical argument is as follows. Because the inflow and outflow rates are equal (both are .001 $\mathrm{m}^3/\mathrm{min}$), $C(t)$ will be increasing as long as the carbon monoxide concentration of the inflow is greater than that of the outflow, which is the concentration in the room. This is certainly the case initially, when the room is free of carbon monoxide. It will remain the case for all time because as the concentration in the room increases toward that of the inflow, the rate at which it increases will decrease toward zero.

(b) What is the behavior of $C(t)$ as $t \to \infty$? (Give reasoning.)

Solution: The first argument given for part (a) already shows that $C(t)$ is an increasing function of t that will approach the equilibrium value of 2 m^3 as $t \to \infty$.

Alternatively, the second argument given for part (a) already shows that as $t \to \infty$ the concentration of carbon monoxide in the room approaches that of the inflow, which is 4%. Because the volume of the room is 50 m^3, this means that $C(t)$ will approach $.04 \cdot 50 = 2$ m^3 as $t \to \infty$.

(c) Derive a formula for $C(t)$.

Solution: The carbon monoxide concentration in the room at time t is $C(t)/50$. Because this is also the concentration of the outflow, $C(t)$, the volume of carbon monoxide in the room, will satisfy

$$\frac{dC}{dt} = \tfrac{1}{1000} \cdot \tfrac{4}{100} - \tfrac{1}{1000} \cdot \frac{C}{50}\,, \qquad C(0) = 0\,.$$

The above differential equation is linear, so we write it as

$$\frac{dC}{dt} + \tfrac{1}{50000}\,C = \tfrac{1}{25000}\,.$$

One sees that an integrating factor is $e^{\frac{1}{50000}t}$, whereby the equation can be recast as

$$\frac{d}{dt}\Big(e^{\frac{1}{50000}t}C\Big) = \tfrac{1}{25000}\,e^{\frac{1}{50000}t}\,.$$

This is then integrated to obtain

$$e^{\frac{1}{50000}t}C = 2e^{\frac{1}{50000}t} + D\,.$$

The value of the integration constant D is found through the initial condition $C(0) = 0$ by setting $t = 0$ and $C = 0$, whereby

$$D = e^0 0 - 2e^0 = -2\,.$$

Then solving for C gives

$$C(t) = 2 - 2e^{-\frac{1}{50000}t}\,.$$

Remark: I had originally intended that the problem would state that the cigarette smoke contains .4% carbon monoxide, a more realistic value.

(4) (6 points) Suppose you are using the improved Euler method to numerically solve an initial-value problem over the interval [0,20]. By what factor would you expect the error to decrease when you increase the number of steps taken from 500 to 2000?

Solution: The global error for the improved Euler method is second order. This means that the error scales as h^2, where h is the time step. When the number of steps taken increases from 500 to 2000, the time step h decreases by a factor of 4. The error will therefore decrease like h^2 — namely, by a factor of $4^2 = 16$.

(5) (18 points) Give an implicit general solution to each of the following differential equations.

(a) $(3x^2 - 2xy)\,dx + (6y^2 - x^2)\,dy = 0\,.$

Solution: Because

$$\partial_y(3x^2 - 2xy) = -2x \quad = \quad \partial_x(6y^2 - x^2) = -2x\,,$$

the equation is *exact*. Hence, we can find $H(x,y)$ such that

$$\partial_x H = 3x^2 - 2xy\,, \qquad \partial_y H = 6y^2 - x^2\,.$$

The first of these shows that

$$H(x,y) = x^3 - x^2 y + h(y)\,.$$

Plugging this into the second then yields

$$6y^2 - x^2 = \partial_y H = -x^2 + h'(y)\,.$$

Hence, $h'(y) = 6y^2$, or $h(y) = 2y^3$. The general solution is therefore given implicitly by

$$x^3 - x^2 y + 2y^3 = C\,,$$

where C is an arbitrary constant.

(b) $\left(\dfrac{y}{x} + 3x\right) dx + dy = 0\,.$

Solution: Because

$$\partial_y\left(\frac{y}{x} + 3x\right) = \frac{1}{x} \quad \neq \quad \partial_x 1 = 0\,,$$

the equation is *not exact*. Seek an integrating factor $\mu(x,y)$ such that

$$\partial_y\left[\left(\frac{y}{x} + 3x\right)\mu\right] = \partial_x \mu\,.$$

This means that μ must satisfy

$$\left(\frac{y}{x} + 3x\right)\partial_y \mu + \frac{1}{x}\mu = \partial_x \mu\,.$$

If we assume that μ depends only on x (so that $\partial_y \mu = 0$) then this reduces to

$$\frac{1}{x}\mu = \partial_x \mu\,.$$

From this one obtains an integrating factor as

$$\mu(x) = \exp\left(\int \frac{1}{x}\,dx\right) = \exp\big(\ln(x)\big) = x\,.$$

This implies that

$$(y + 3x^2)\, dx + x dy = 0 \quad \text{is exact.}$$

(If you had seen this more directly, all the better.) You can therefore find $H(x, y)$ such that

$$\partial_x H = (y + 3x^2)\,, \qquad \partial_y H = x\,.$$

The first of these implies that

$$H(x, y) = xy + x^3 + h(y)\,.$$

Plugging this into the second equation yields

$$x = \partial_y H = x + h'(y)\,,$$

Hence, $h'(y) = 0$, or $h(y) = 0$. The general solution is therefore given implicitly by

$$xy + x^3 = C\,,$$

where C is an arbitrary constant.

Remark: An alternative approach to this problem begins by noticing that it is linear in y with the normal form

$$\frac{dy}{dx} + \frac{1}{x}\, y = -3x^2\,.$$

(6) (16 points) A 9 kilogram (kg) mass initially at rest is dropped in a medium that offers a resistance force of $v^2/20$ newtons (= kg m/sec^2) where v is the downward velocity of the mass in meters per second. The gravitational acceleration is 9.8 m/sec^2.

(a) What is the terminal velocity of the mass?

Solution: The terminal velocity is the velocity at which the force of resistance balances that of gravity. This happens when

$$\tfrac{1}{20} v^2 = mg = 9 \cdot 9.8\,.$$

Upon solving for v one obtains

$$\begin{aligned} v &= \sqrt{20 \cdot 9 \cdot 9.8} \text{ m/sec} \qquad \text{(full marks)} \\ &= \sqrt{9 \cdot 2 \cdot 98} \\ &= \sqrt{9 \cdot 4 \cdot 49} \\ &= 3 \cdot 2 \cdot 7 = 42 \text{ m/sec}\,. \end{aligned}$$

(b) Write down an initial value problem that governs v as a function of time. (You do not have to solve it!)

Solution: The net downward force on the mass is the force of gravity minus the force of resistance. By Newton ($ma = F$), this leads to

$$m \frac{dv}{dt} = mg - \tfrac{1}{20} v^2\,.$$

Because $m = 9$ and $g = 9.8$ and the mass is initially at rest, this yields the initial-value problem

$$\frac{dv}{dt} = 9.8 - \tfrac{1}{180} v^2\,, \qquad v(0) = 0\,.$$

(7) (12 points) Consider the following MATLAB function M-file.

```
function [t,y] = solveit(ti, yi, tf, n)

h = (tf – ti)/n;
t = zeros(n + 1, 1);
y = zeros(n + 1, 1);
t(1) = ti;
y(1) = yi;
for i = 1:n
t(i + 1) = t(i) + h;
y(i + 1) = y(i) + h*(y(i) - y(i)^3);
end
```

(a) What is the initial-value problem being solved numerically?

Solution: It is solving the initial-value problem

$$\frac{dy}{dt} = y - y^3\,, \qquad y = \text{yi at } t = \text{ti}\,, \qquad \text{over [ti,tf]}\,.$$

(b) What is the numerical method being used to solve it?

Solution: It is using the (forward) Euler method.

(c) What are the output values of t(2) and y(2) that you would expect for input values of ti = 0, yi = .5, tf = 5, n = 20 for this method?

Solution: Because it takes 20 timesteps over the inteval $[0, 5]$, the time step h will be given by

$$\text{h} = \frac{5-0}{20} = \frac{5}{20} = \frac{1}{4} = .25\,.$$

Because t(1) = ti = 0 and h = .25, one sees that t(2) will be given by

$$\text{t}(2) = \text{t}(1) + \text{h} = 0 + .25 = .25\,.$$

Because y(1) = yi = .5 and h = .25, one sees that y(2) will be given by

$$\begin{aligned}\text{y}(2) = \text{y}(1{+}1) &= \text{y}(1) + \text{h}\,(\text{y}(1) - \text{y}(1)^3)\\ &= \tfrac{1}{2} + \tfrac{1}{4}\,(\tfrac{1}{2} - (\tfrac{1}{2})^3) \qquad \text{(full marks)}\\ &= \tfrac{1}{2} + \tfrac{1}{4}\,\tfrac{3}{8}\\ &= \tfrac{1}{2} + \tfrac{3}{32} = \tfrac{19}{32}\,,\end{aligned}$$

or alternatively,

$$\begin{aligned}\text{y}(2) = \text{y}(1{+}1) &= \text{y}(1) + \text{h}\,(\text{y}(1) - \text{y}(1)^3)\\ &= .5 + .25\,(.5 - (.25)^3) \qquad \text{(full marks)}\\ &= .5 + .25\,(.5 - .125)\\ &= .5 + .25\,.375 = .59375\,.\end{aligned}$$