Math 246, Second In-Class Exam (Wednesday, 9 April 2003)

Professor Levermore

Before doing anything else, on the cover of your examination booklet please print your name and student identification number. Then print the UMD Honor Pledge and sign your name under it:

I pledge on my honor that I have not given or received any unauthorized assistance on this examination.

Exam ends at 5 minutes of the hour, no later. Exam is closed book and closed notes. No calculators or other electronic devices are allowed. Work at most one problem on each page of your examination booklet, marking the problem number at the top of the page. Your answer(s) to each part should be circled, and you must show your work. Brief descriptions of your reasoning are helpful, especially for earning partial credit. Any work that you do not want to be considered should be crossed out.

(1) (12 points) Let L be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (shown with multiplicities) are $-3+i4$, $-3+i4$, $-3-i4$, $-3-i4$, $i5$, $-i5$, -7, -7, 0, 0.

(a) What is the order of L?

(b) Give a general real solution of the homogeneous equation $Ly = 0$?

(2) (9 points) Solve the initial-value problem

$$y'' - 6y' + 9y = 0\,, \qquad y(0) = 0\,, \quad y'(0) = 1\,.$$

(3) (27 points) Find a general solution for each of the following equations.

(a) $y'' + 16y = 5e^{3t}$.

(b) $y'' + 4y' + 8y = 6\sin(2t)$.

(c) $y'' + 2y' - 3y = e^{t}$.

(4) (9 points) The functions $1 + x$ and e^x are solutions of the equation

$$xy'' - (1+x)y' + y = 0\,, \qquad x > 0\,.$$

(You do not have to check that this is true.)

(a) Compute their Wronskian.

(b) Find a general solution of the equation

$$xy'' - (1+x)y' + y = x^2 e^x\,, \qquad x > 0\,.$$

(5) (6 points) The vertical displacement of a mass on a spring is given by

$$z(t) = 4\cos(7t) + 3\sin(7t)\,.$$

Express this in the form $z(t) = A\cos(\omega t - \delta)$, identifying the amplitude and phase of the oscillation.

(6) (10 points) When a 2 kilogram (kg) mass is hung vertically from a spring, at rest it stretches the spring .2 meters (m). (Gravitational acceleration is $g = 9.8$ m/sec^2.) At $t = 0$ the mass is displaced .1 m above its equilibrium position and released with no initial velocity. It moves in a medium that imparts a drag force of 4 Newtons (1 Newton = 1 kg m/sec^2) when the speed of the mass is 5 m/sec. There are no other forces. (As usual, assume the spring force is proportional to displacement and the drag force is proportional to velocity.)

  (a) Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve the initial-value problem, just write it down!)
  (b) Give the natural frequency of the spring.
  (c) Show that the system is under-damped and give its quasifrequency.

(7) (6 points) Compute the Laplace transform of $f(t) = e^{-4t}$ from its definition.

(8) (9 points) Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y'' + 9y = f(t)\,, \qquad y(0) = 4\,, \quad y'(0) = 1\,,$$

where

$$f(t) = \begin{cases} 0 & \text{for } 0 \le t < 2\pi\,, \\ t - 2\pi & \text{for } t \ge 2\pi\,. \end{cases}$$

You may refer to the table below. (DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$.)

(9) (12 points) Find the inverse Laplace transform of the following functions:

  (a) $F(s) = \dfrac{4s}{s^2 - 4}$,

  (b) $F(s) = \dfrac{6se^{-5s}}{s^2 + 9}$.

You may refer to the table below.

A Short Table of Laplace Transforms

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad \text{for } s > 0\,.$$

$$\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2 + b^2} \qquad \text{for } s > 0\,.$$

$$\mathcal{L}\{\sin(bt)\} = \frac{b}{s^2 + b^2} \qquad \text{for } s > 0\,.$$

$$\mathcal{L}\{e^{at} f(t)\} = F(s - a) \qquad \text{where } F(s) = \mathcal{L}\{f(t)\}\,.$$

$$\mathcal{L}\{u(t - c) f(t - c)\} = e^{-cs} F(s) \qquad \text{where } F(s) = \mathcal{L}\{f(t)\} \text{ and } u \text{ is the step function}\,.$$