Math 246, Second In-Class Exam Solutions (Spring 2003)
Professor Levermore

(1) (12 points) Let L be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (shown with multiplicities) are $-3+i4$, $-3+i4$, $-3-i4$, $-3-i4$, $i5$, $-i5$, -7, -7, 0, 0.

(a) What is the order of L?

Solution: There are ten roots listed, so the degree of the characteristic polynomial is ten, and consequently the order of L must be ten.

(b) Give a general real solution of the homogeneous equation $Ly = 0$?

Solution: The general solution is

$$y = c_1 e^{-3t}\cos(4t) + c_2 e^{-3t}\sin(4t) + c_3 t\, e^{-3t}\cos(4t) + c_4 t\, e^{-3t}\sin(4t) \\ + c_5\cos(5t) + c_6\sin(5t) + c_7 e^{-7t} + c_8 t\, e^{-7t} + c_9 + c_{10} t\,.$$

The reasoning is as follows.

- The double conjugate pair $-3 \pm i4$ yields

$$e^{-3t}\cos(4t)\,, \quad e^{-3t}\sin(4t)\,, \quad t\, e^{-3t}\cos(4t)\,, \quad \text{and} \quad t\, e^{-3t}\sin(4t)\,.$$

- The conjugate pair $\pm i5$ yields $\cos(5t)$ and $\sin(5t)$.
- The double real root -7 yields e^{-7t} and $t\, e^{-7t}$.
- The double real root 0 yields 1 and t.

(2) (9 points) Solve the initial-value problem

$$y'' - 6y' + 9y = 0\,, \qquad y(0) = 0\,, \quad y'(0) = 1\,.$$

Solution: This is a constant coefficient, homogeneous linear initial-value problem. It may be either (1) solved by first finding the general solution or (2) solved directly using the Laplace transform.

The characteristic polynomial is

$$P(z) = z^2 - 6z + 9 = (z-3)^2\,.$$

It has the double real root 3, which yields the general solution

$$y(t) = c_1 e^{3t} + c_2 t\, e^{3t}\,.$$

Because

$$y'(t) = 3c_1 e^{3t} + c_2(e^{3t} + 3t\, e^{3t})\,,$$

when the initial conditions are imposed, one finds that

$$y(0) = c_1 = 0\,, \qquad y'(0) = 3c_1 + c_2 = 1\,.$$

These are solved to find that $c_1 = 0$ and $c_2 = 1$. The solution of the initial-value problem is therefore

$$y(t) = t\, e^{3t}\,.$$

We now show how to solve the problem using the Laplace transform. The Laplace transform of the initial-value problem is

$$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = 0\,,$$

where

$$\begin{aligned}\mathcal{L}\{y\}(s) &= Y(s)\,,\\ \mathcal{L}\{y'\}(s) &= s\,Y(s) - y(0) = s\,Y(s)\,,\\ \mathcal{L}\{y''\}(s) &= s^2Y(s) - s\,y(0) - y'(0) = s^2Y(s) - 1\,.\end{aligned}$$

The Laplace transform of the initial-value problem then becomes

$$s^2Y(s) - 1 - 6s\,Y(s) + 9Y(s) = 0\,,$$

which can be put in the form

$$(s^2 - 6s + 9)Y(s) = 1\,.$$

Upon solving this for $Y(s)$ one finds that

$$Y(s) = \frac{1}{s^2 - 6s + 9} = \frac{1}{(s-3)^2}\,.$$

Referring to the table on the last page, Item 1 with $n = 1$ gives $\mathcal{L}\{t\} = 1/s^2$. Item 4 with $a = 3$ and $f(t) = t$ then gives

$$\mathcal{L}\{e^{3t}t\} = \frac{1}{(s-3)^2}\,.$$

One thereby concludes that

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}(t) = t\,e^{3t}\,.$$

(3) (27 points) Find a general solution for each of the following equations.

(a) $y'' + 16y = 5e^{3t}$.

Solution: This is a constant coefficient, nonhomogeneous linear problem. The characteristic polynomial of its homogeneous part is

$$P(z) = z^2 + 16\,.$$

It has the simple complex pair of roots $-i4$ and $i4$, which yields the general homogeneous solution

$$y_H(t) = c_1\cos(4t) + c_2\sin(4t)\,.$$

Because the forcing is of the form e^{zt} for $z = 3$, and because $z = 3$ is not a root of the characteristic polynomial, a particular solution can be found quickly by either the method of undetermined coefficients (as in the book) or the method of determined coefficients (as in class).

The method of *undetermined coefficients* seeks a particular solution of the form $y_P(t) = Ae^{3t}$. Because

$$y_P(t) = Ae^{3t}\,, \qquad y_P'(t) = 3Ae^{3t}\,, \qquad y_P''(t) = 9Ae^{3t}\,,$$

one sees that

$$Ly_P = y_P'' + 16y_P = (9 + 16)Ae^{3t} = 25Ae^{3t} = 5e^{3t}\,.$$

which implies $A = 1/5$. Hence, $y_P(t) = \frac{1}{5}e^{3t}$.

The method of *determined coefficients* evaluates the identity

$$L(e^{zt}) = (z^2 + 16)\, e^{zt}\,,$$

at $z = 3$ to obtain $L(e^{3t}) = 25e^{3t}$. Dividing this by 5 gives $L(\frac{1}{5}e^{3t}) = 5e^{3t}$, which shows that $y_P(t) = \frac{1}{5}e^{3t}$.

By either method you find the same y_P, and the general solution of the problem is therefore

$$y(t) = c_1 \cos(4t) + c_2 \sin(4t) + \tfrac{1}{5}e^{3t}\,.$$

(b) $y'' + 4y' + 8y = 6\sin(2t)\,.$

Solution: This is a constant coefficient, nonhomogeneous linear problem. The characteristic polynomial of its homogeneous part is

$$P(z) = z^2 + 4z + 8 = (z+2)^2 + 4\,.$$

It has the conjugate pair of roots $-2 \pm i2$, which yields the general homogeneous solution

$$y_H(t) = c_1 e^{-2t}\cos(2t) + c_2 e^{-2t}\sin(2t)\,.$$

Because the forcing is of the form e^{zt} for $z = i2$, and because $z = i2$ is not a root of the characteristic polynomial, a particular solution can be found quickly by either the method of undetermined coefficients (as in the book) or the method of determined coefficients (as in class).

The method of *undetermined coefficients* seeks a particular solution of the form $y_P(t) = A\cos(2t) + B\sin(2t)$. Because

$$\begin{aligned} y_P(t) &= A\cos(2t) + B\sin(2t)\,,\\ y_P'(t) &= -2A\sin(2t) + 2B\cos(2t)\,,\\ y_P''(t) &= -4A\cos(2t) - 4B\sin(2t)\,, \end{aligned}$$

one sees that

$$\begin{aligned} Ly_P &= y_P'' + 4y_P' + 8y_P\\ &= (4A + 8B)\cos(2t) + (4B - 8A)\sin(2t) = 6\sin(2t)\,. \end{aligned}$$

This leads to the algebraic linear system

$$4A + 8B = 0\,, \qquad 4B - 8A = 6\,.$$

This can be solved to find that $A = -3/5$ and $B = 3/10$. Hence,

$$y_P(t) = -\frac{3}{5}\cos(2t) + \frac{3}{10}\sin(2t)\,.$$

The method of *determined coefficients* evaluates the KEY identity $Le^{zt} = (z^2 + 4z + 8)e^{zt}$ at $z = i2$ to obtain $Le^{i2t} = (4 + i8)e^{i2t}$. Multiplying this by $6/(4 + i8)$ shows that

$$L\left(\frac{6}{4+i8}e^{i2t}\right) = 6e^{i2t}\,.$$

Because $6e^{i2t} = 6\cos(2t) + i6\sin(2t)$, the imaginary part of the left-hand side above will be Ly_{P}. Because

$$\begin{aligned}\frac{6}{4+i8}\,e^{i2t} &= \frac{6}{4+i8}\,\frac{4-i8}{4-i8}\,e^{i2t} = \frac{6(4-i8)}{4^2+8^2}\,e^{i2t} \\ &= \frac{6(4-i8)}{80}\big(\cos(2t) + i\sin(2t)\big) \\ &= \left(\frac{24}{80}\cos(2t) + \frac{48}{80}\sin(2t)\right) \\ &\quad + i\left(-\frac{48}{80}\cos(2t) + \frac{24}{80}\sin(2t)\right),\end{aligned}$$

this imaginary part shows that

$$y_{P}(t) = -\frac{48}{80}\cos(2t) + \frac{24}{80}\sin(2t) = -\frac{3}{5}\cos(2t) + \frac{3}{10}\sin(2t)\,.$$

By either method you find the same y_{P}, and the general solution of the problem is therefore

$$y = c_1 e^{-2t}\cos(2t) + c_2 e^{-2t}\sin(2t) - \frac{3}{5}\cos(2t) + \frac{3}{10}\sin(2t)\,.$$

(c) $y'' + 2y' - 3y = e^t$.

Solution: This is a constant coefficient, nonhomogeneous linear problem. The characteristic polynomial of its homogeneous part is

$$P(z) = z^2 + 2z - 3 = (z-1)(z+3)\,.$$

It the simple real roots -3 and 1, which yields the general homogeneous solution

$$y_{H}(t) = c_1 e^{-3t} + c_2 e^t\,.$$

Because the forcing is of the form e^{zt} for $z = 1$, and because $z = 1$ is a root of the characteristic polynomial, a particular solution can be found quickly by either the method of undetermined coefficients (as in the book) or the method of determined coefficients (as in class).

The method of *undetermined coefficients* seeks a particular solution of the form $y_{P}(t) = Ate^t$. Because

$$y_{P}'(t) = A(e^t + t\,e^t)\,, \qquad y_{P}''(t) = A(2e^t + t\,e^t)\,,$$

one sees that

$$\begin{aligned}Ly_{P} &= y_{P}'' + 2y_{P}' - 3y_{P} \\ &= A(2e^t + t\,e^t) + 2A(e^t + t\,e^t) - 3t\,e^t \\ &= A4e^t = e^t\,,\end{aligned}$$

which implies $A = 1/4$. Hence, $y_{P}(t) = \frac{1}{4}te^t$.

The method of *determined coefficients* evaluates the identity

$$L(t\,e^{zt}) = (z^2 + 2z - 3)\,t\,e^{zt} + (2z+2)\,e^{zt}\,,$$

at $z = 1$ to obtain $L(te^t) = 4e^t$. Dividing this by 4 gives $L(\frac{1}{4}t\,e^t) = e^t$, which shows that $y_{P}(t) = \frac{1}{4}t\,e^t$.

By either method you find the same y_P, and the general solution of the problem is therefore

$$y(t) = c_1 e^{-t} + c_2 e^t + \tfrac{1}{4} t\, e^t \,.$$

(4) (9 points) The functions $1 + x$ and e^x are solutions of the equation

$$xy'' - (1 + x)y' + y = 0 \,, \qquad x > 0 \,.$$

(You do not have to check that this is true.)

(a) Compute their Wronskian.

Solution: The Wronskian $W(x)$ of $1 + x$ and e^x is given by

$$W(x) = \det\begin{pmatrix} 1+x & e^x \\ 1 & e^x \end{pmatrix} = (1 + x)\, e^x - e^x = x\, e^x \,.$$

Note $W(x) > 0$ when $x > 0$, so $1 + x$ and e^x are linearly independent.

(b) Find a general solution of the equation

$$xy'' - (1 + x)y' + y = x^2 e^x \,, \qquad x > 0 \,.$$

Solution: The general solution of this nonhomogeneous equation will have the form $y = y_H + y_P$ where y_H is the general solution of the corresponding homogeneous equation and y_P is any particular solution of the nonhomogeneous equation. Because you are given that $1 + x$ and e^x are solutions of the corresponding homogeneous equation, and you know by part (a) that they are linearly independent, you know that

$$y_H = c_1(1 + x) + c_2 e^x \,.$$

The methods of undetermined or determined coefficients cannot be used to find a particular solution, so we will use the method of variation of parameters. We first put the equation into its normal form

$$y'' - \frac{1+x}{x}\, y' + \frac{1}{x}\, y = x\, e^x \,,$$

and then seek y_P of the form

$$y_P = (1 + x)u_1(x) + e^x u_2(x) \,.$$

One chooses u_1' and u_2' so that they satisfy

$$(1 + x)u_1' + e^x u_2' = 0 \,, \qquad u_1' + e^x u_2' = x\, e^x \,.$$

This linear system is solved to find that

$$u_1' = -e^x \,, \qquad u_2' = 1 + x \,.$$

Upon integrating these, you find that

$$u_1(x) = c_1 - e^x \,, \qquad u_2(x) = c_2 + x + \tfrac{1}{2} x^2 \,.$$

Your answer can be expressed as

$$y = c_1(1 + x) + c_2 e^x - (1 + x)e^x + e^x(x + \tfrac{1}{2} x^2) \,.$$

This can be simplified to

$$y = c_1(1 + x) + c_3 e^x + \tfrac{1}{2} x^2 e^x \,,$$

where $c_3 = c_2 - 1$.

(5) (6 points) The vertical displacement of a mass on a spring is given by

$$z(t) = 4\cos(7t) + 3\sin(7t)\,.$$

Express this in the form $z(t) = A\cos(\omega t - \delta)$, identifying the amplitude and phase of the oscillation.

Solution: The displacement takes the form

$$z(t) = 5\cos\left(7t - \tan^{-1}(\tfrac{3}{4})\right),$$

where the amplitude is 5, the frequency is 7, and the phase is $\tan^{-1}(\frac{3}{4})$. There are several approaches to this problem. Here are two.

One approach that requires no memorization other than the usual addition formula for cosine is as follows. Because

$$A\cos(\omega t - \delta) = A\cos(\delta)\cos(\omega t) + A\sin(\delta)\sin(\omega t)\,,$$

this form will be equal to $y(t)$ provided $\omega = 7$ and

$$A\cos(\delta) = 4\,, \qquad A\sin(\delta) = 3\,.$$

Upon solving these equations one finds that the amplitude A is given by

$$A = \sqrt{(4^2 + 3^2} = \sqrt{16+9} = \sqrt{25} = 5\,,$$

while the phase δ is given either by

$$\delta = \sin^{-1}\left(\frac{3}{A}\right) = \sin^{-1}\left(\frac{3}{5}\right),$$

or by

$$\delta = \cos^{-1}\left(\frac{4}{A}\right) = \cos^{-1}\left(\frac{4}{5}\right),$$

or by

$$\delta = \tan^{-1}\left(\frac{3}{4}\right).$$

Another approach requires you to memorize special formulas for both the amplitude and phase of functions of the form

$$c_1\cos(\omega t) + c_2\sin(\omega t)\,.$$

The formula for the amplitude is easier one because c_1 and c_2 appear in it symmetrically. It gives

$$A = \sqrt{c_1^2 + c_2^2} = \sqrt{4^2 + 3^2} = \sqrt{16+9} = \sqrt{25} = 5\,.$$

The formula for the phase is trickier because c_1 and c_2 do not appear in it symmetrically. It gives

$$\delta = \tan^{-1}\left(\frac{c_2}{c_1}\right) = \tan^{-1}\left(\frac{3}{4}\right).$$

The most common mistake made by those who chose this approach was to exchange the roles of c_1 and c_2 in this formula. One way to keep these roles straight is to remember the formula verbally as

$$\text{phase} = \tan^{-1}\left(\frac{\text{coefficient of sine}}{\text{coefficient of cosine}}\right).$$

(6) (10 points) When a 2 kilogram (kg) mass is hung vertically from a spring, at rest it stretches the spring .2 meters (m). (Gravitational acceleration is $g = 9.8$ m/sec^2.) At $t = 0$ the mass is displaced .1 m above its equilibrium position and released with no initial velocity. It moves in a medium that imparts a drag force of 4 Newtons (1 Newton = 1 kg m/sec^2) when the speed of the mass is 5 m/sec. There are no other forces. (As usual, assume the spring force is proportional to displacement and the drag force is proportional to velocity.)

(a) Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve the initial-value problem, just write it down!)

Solution: Let y be the displacement of the mass from the equilibrium position in meters, with upward displacements being positive. The governing initial-value problem then has the form

$$my'' + \gamma y' + ky = 0\,, \qquad y(0) = .1\,, \quad y'(0) = 0\,,$$

where m is the mass, γ is the drag coefficient, and k is the spring constant. The problem says that $m = 2$ kilograms. The spring constant is obtained by balancing the weight of the mass ($mg = 2 \cdot 9.8$ Newtons) with the force applied by the spring when it is stretched .2 meters. This gives $.2k = 2 \cdot 9.8$, or

$$k = \frac{2 \cdot 9.8}{.2} = 10 \cdot 9.8 = 98 \quad \text{kg/sec}^2\,.$$

The drag coefficient is obtained by balancing the force of 4 Newtons with the drag force imparted by the medium when the speed of the mass is 5 m/sec. This gives $\gamma 5 = 4$, or

$$\gamma = \frac{4}{5} \qquad \text{kg/sec}\,.$$

The governing initial-value problem is therefore

$$2y'' + \frac{4}{5}y' + 98y = 0\,, \qquad y(0) = .1\,, \quad y'(0) = 0\,.$$

If you had chosen downward displacements to be positive then the governing initial-value problem would be identical except for the first initial condition, which would then be $y(0) = -.1$.

(b) Give the natural frequency of the spring.

Solution: The natural frequency of the spring is given by

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{98}{2}} = \sqrt{49} = 7 \quad 1/\text{sec}\,.$$

(c) Show that the system is under-damped and give its quasifrequency.

Solution: The characteristic polynomial is

$$P(z) = z^2 + \frac{2}{5}z + 49 = \left(z + \frac{1}{5}\right)^2 + 49 - \frac{1}{5^2}\,,$$

which has the complex roots

$$z = -\tfrac{1}{5} \pm i\sqrt{49 - \tfrac{1}{25}}\,.$$

The system is therefore under-damped with a quasifrequency μ given by

$$\mu = \sqrt{49 - \tfrac{1}{25}}\,.$$

(7) (6 points) Compute the Laplace transform of $f(t) = e^{-4t}$ from its definition.
Solution: Let $F(s) = \mathcal{L}\{f\}(s)$. By the definition of the Laplace transform

$$F(s) \equiv \int_0^{\infty} e^{-st}e^{-4t}\,dt = \lim_{M\to\infty} \int_0^{M} e^{-(s+4)t}\,dt\,.$$

For $s+4 \neq 0$ one has

$$\int_0^{M} e^{-(s+4)t}\,dt = \left(-\frac{e^{-(s+4)t}}{s+4}\right)\Bigg|_0^M = \left[-\frac{e^{-(s+4)M}}{s+4} + \frac{1}{s+4}\right],$$

while for $s+4 = 0$ one has

$$\int_0^{M} e^{-(s+4)t}\,dt = \int_0^{M} 1\,dt = M\,.$$

one thereby sees that

$$\begin{aligned} F(s) &= \lim_{M\to\infty} \begin{cases} \left[-\dfrac{e^{-(s+4)M}}{s+4} + \dfrac{1}{s+4}\right] & \text{for } s+4 \neq 0 \\ M & \text{for } s+4 = 0 \end{cases} \\ &= \begin{cases} \dfrac{1}{s+4} & \text{for } s+4 > 0 \\ \text{diverges} & \text{for } s+4 \leq 0\,. \end{cases} \end{aligned}$$

Hence, one finds that

$$\mathcal{L}\{e^{-4t}\}(s) = \frac{1}{s+4} \qquad \text{for } s > -4\,.$$

(8) (9 points) Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y'' + 9y = f(t)\,, \qquad y(0) = 4\,, \quad y'(0) = 1\,,$$

where

$$f(t) = \begin{cases} 0 & \text{for } 0 \leq t < 2\pi\,, \\ t - 2\pi & \text{for } t \geq 2\pi\,. \end{cases}$$

You may refer to the table below. (DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$.)
Solution: The Laplace transform of the initial-value problem is

$$\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \mathcal{L}\{f\}\,,$$

where

$$\begin{aligned} \mathcal{L}\{y\} &= Y(s)\,, \\ \mathcal{L}\{y'\} &= sY(s) - y(0) = sY(s) - 4\,, \\ \mathcal{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s4 - 1\,. \end{aligned}$$

To compute $\mathcal{L}\{f\}$, first rewrite f as

$$f(t) = u(t - 2\pi)\,(t - 2\pi)\,.$$

Referring to the table on the last page, Item 5 with $c = 2\pi$ and $f(t) = t$ followed Item 1 with $n = 1$ then shows that

$$\begin{aligned}\mathcal{L}\{f\} &= \mathcal{L}\{u(t-2\pi)\,(t-2\pi)\}\\ &= e^{-2\pi s}\mathcal{L}\{t\}(s)\\ &= e^{-2\pi s}\,\frac{1}{s^2}\,.\end{aligned}$$

The Laplace transform of the initial-value problem then becomes

$$\big(s^2Y(s) - 4s - 1\big) + 9Y(s) = e^{-2\pi s}\,\frac{1}{s^2}\,,$$

which becomes

$$(s^2+9)Y(s) - (4s+1) = e^{-2\pi s}\,\frac{1}{s^2}\,.$$

Hence, $Y(s)$ is given by

$$Y(s) = \frac{1}{s^2+9}\left(4s + 1 + e^{-2\pi s}\,\frac{1}{s^2}\right).$$

(9) (12 points) Find the inverse Laplace transform of the following functions:

(a) $F(s) = \dfrac{4s}{s^2-4}$,

Solution: The denominator factors as $(s-2)(s+2)$ so the partial fraction decomposition is

$$F(s) = \frac{4s}{s^2-4} = \frac{4s}{(s-2)(s+2)} = \frac{2}{s-2} + \frac{2}{s+2}\,.$$

Referring to the table on the last page, Item 1 with $n = 0$ gives $\mathcal{L}\{1\} = 1/s$. Item 4 with $a = 2$, $a = -2$ and $f(t) = 1$ then gives

$$\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}\,, \qquad \mathcal{L}\{e^{-2t}\} = \frac{1}{s+2}\,.$$

One therefore finds that

$$\mathcal{L}^{-1}\left\{\frac{4s}{s^2-4}\right\} = 2e^{2t} + 2e^{-2t}\,.$$

(b) $F(s) = \dfrac{6se^{-5s}}{s^2+9}$.

Solution: Referring to the table on the last page, Item 2 with $b = 3$ gives

$$\mathcal{L}\{\cos(3t)\} = \frac{s}{s^2+9}\,.$$

Item 5 with $c = 5$ and $f(t) = 6\cos(3t)$ then gives

$$\mathcal{L}\{u(t-5)\,6\cos(3(t-5))\} = e^{-5s}\,\frac{6s}{s^2+9}\,.$$

One therefore finds that

$$\mathcal{L}^{-1}\left\{\frac{6se^{-5s}}{s^2+9}\right\} = u(t-5)\,6\cos(3(t-5))\,.$$

A Short Table of Laplace Transforms

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad \text{for } s > 0\,.$$

$$\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2} \qquad \text{for } s > 0\,.$$

$$\mathcal{L}\{\sin(bt)\} = \frac{b}{s^2+b^2} \qquad \text{for } s > 0\,.$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \qquad \text{where } F(s) = \mathcal{L}\{f(t)\}\,.$$

$$\mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs}F(s) \qquad \text{where } F(s) = \mathcal{L}\{f(t)\}$$

and u is the step function.