Math 246, Third In-Class Exam (Wednesday, 7 May 2003)

Professor Levermore

Before doing anything else, on the cover of your examination booklet please print your name and student identification number. Then print the UMD Honor Pledge and sign your name under it:

I pledge on my honor that I have not given or received any unauthorized assistance on this examination.

Exam ends at 5 minutes of the hour, no later. Exam is closed book and closed notes. No calculators or other electronic devices are allowed. Work at most one problem on each page of your examination booklet, marking the problem number at the top of the page. Your answer(s) to each part should be circled, and you must show your work. Brief descriptions of your reasoning are helpful, especially for earning partial credit. Any work that you do not want to be considered should be crossed out.

(1) (12 points) Consider the matrices

$$A = \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix}, \qquad B = \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix}.$$

Compute the matrices

(a) A^T,
(b) $\overline{A}$,
(c) A^*,
(d) $5A - B$,
(e) AB,
(f) B^{-1}.

(2) (8 points) Consider the matrix

$$A = \begin{pmatrix} 3 & 3 \\ 4 & -1 \end{pmatrix}.$$

(a) Find all the eigenvalues of A.
(b) For each eigenvalue of A find an eigenvector.

(3) (10 points) Consider the linear algebraic system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 2\,, \\ 2x_1 + x_2 + x_3 &= 1\,, \\ x_1 - x_2 + 2x_3 &= -1\,. \end{aligned}$$

Either find its general solution or else show that it has no solution.

(4) (20 points) Solve each of the following initial-value problems.

(a) $$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(b) $$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(5) (20 points) Find a greneral solution for each of the following systems.

(a) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(b) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(6) (12 points) Sketch phase-plane portraits for each of the following systems. State the type and stability of the origin. (Notice that these systems also appear in Problems 4 and 5.)

(a) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(b) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(c) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(7) (9 points) Consider the system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - xy \\ 4y - xy - 2y^2 \end{pmatrix}.$$

(a) Find all of its equilibrium points.
(b) Compute the coefficient matrix of the linearization associated with each equilibrium point.

(8) (9 points) Suppose you know that for some nonlinear system of differential equations

- the equilibrium solutions are $(0, 0)$, $(4, -2)$, and $(4, 2)$;
- for $(0, 0)$ the linearization has eigenvalues -2 and -1 with respective eigenvectors

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix};$$

- for $(4, -2)$ the linearization has eigenvalues 2 and 1 with respective eigenvectors

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

- for $(4, 2)$ the linearization has eigenvalues 1 and -1 with respective eigenvectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

Sketch a plausible phase portrait for the system. Identify the type and stability of each equilibrium solution.