Math 246, Third In-Class Exam Solutions (Spring 2003)
Professor Levermore

(1) (12 points) Consider the matrices

$$A = \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix}, \qquad B = \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix}.$$

Compute the matrices

(a) A^T,

Solution: The transpose of A is given by

$$A^T = \begin{pmatrix} -i2 & 2+i \\ 1+i & -4 \end{pmatrix}.$$

(b) $\overline{A}$,

Solution: The conjugate of A is given by

$$\overline{A} = \begin{pmatrix} i2 & 1-i \\ 2-i & -4 \end{pmatrix}.$$

(c) A^*,

Solution: The adjoint of A is given by

$$A^* = \begin{pmatrix} -i2 & 2-i \\ 1-i & -4 \end{pmatrix}.$$

(d) $5A - B$,

Solution: The difference of $5A$ and B is given by

$$\begin{aligned} 5A - B &= \begin{pmatrix} -i10 & 5+i5 \\ 10+i5 & -20 \end{pmatrix} - \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} \\ &= \begin{pmatrix} 7-i10 & -1+i5 \\ 2+i5 & -27 \end{pmatrix}. \end{aligned}$$

(e) AB,

Solution: The product of A and B is given by

$$\begin{aligned} AB &= \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix} \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} \\ &= \begin{pmatrix} -i2\cdot 7 + (1+i)\cdot 8 & -i2\cdot 6 + (1+i)\cdot 7 \\ (2+i)\cdot 7 - 4\cdot 8 & (2+i)\cdot 6 - 4\cdot 7 \end{pmatrix} \\ &= \begin{pmatrix} 8-i6 & 7-i5 \\ -18+i7 & -16+i6 \end{pmatrix}. \end{aligned}$$

(f) B^{-1}.

Solution: Observe that it is clear that B has an inverse because

$$\det(B) = \det \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} = 7\cdot 7 - 6\cdot 8 = 49 - 48 = 1 \neq 0.$$

The inverse of B is given by

$$B^{-1} = \begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix}.$$

This may be computed in a number of ways. Here are three.

First, it can be computed by applying elementary row operations to transform the augmented matrix $(B\,|\,I)$ into $(I\,|\,B^{-1})$ as follows:

$$\begin{aligned}(B \quad | \quad I) &= \left(\begin{matrix}7 & 6\\ 8 & 7\end{matrix}\;\middle|\;\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\right) \sim \left(\begin{matrix}1 & \frac{6}{7}\\ 8 & 7\end{matrix}\;\middle|\;\begin{matrix}\frac{1}{7} & 0\\ 0 & 1\end{matrix}\right)\\ &\sim \left(\begin{matrix}1 & \frac{6}{7}\\ 0 & \frac{1}{7}\end{matrix}\;\middle|\;\begin{matrix}\frac{1}{7} & 0\\ -\frac{8}{7} & 1\end{matrix}\right) \sim \left(\begin{matrix}1 & \frac{6}{7}\\ 0 & 1\end{matrix}\;\middle|\;\begin{matrix}\frac{1}{7} & 0\\ -8 & 7\end{matrix}\right)\\ &\sim \left(\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\;\middle|\;\begin{matrix}7 & -6\\ -8 & 7\end{matrix}\right) = \left(I \quad | \quad B^{-1}\right).\end{aligned}$$

Second, because B is a two-by-two matrix, its inverse can be computed directly from the formula

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix} \quad \text{whenever } ad-bc \neq 0\,.$$

This formula is just formula (24) on page 352 of the book specialized to the two-by-two case. When applied to B it yields

$$B^{-1} = \begin{pmatrix}7 & 6\\ 8 & 7\end{pmatrix}^{-1} = \frac{1}{49-48}\begin{pmatrix}7 & -6\\ -8 & 7\end{pmatrix} = \begin{pmatrix}7 & -6\\ -8 & 7\end{pmatrix}.$$

Finally, the inverse of B may be computed directly from its definition by seeking a, b, c, and d such that

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}7 & 6\\ 8 & 7\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}7a+6c & 7b+6d\\ 8a+7c & 8b+7d\end{pmatrix}.$$

Then a and c are found by solving the two-by-two system

$$7a+6c=1\,, \qquad 8a+7c=0\,,$$

which gives $a=7$ and $c=-8$. Similarly b and d are found by solving the two-by-two system

$$7b+6d=0\,, \qquad 8b+7d=1\,,$$

which gives $b=-6$ and $d=7$. You thereby find that

$$B^{-1} = \begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}7 & -6\\ -8 & 7\end{pmatrix}.$$

(2) (8 points) Consider the matrix

$$A = \begin{pmatrix}3 & 3\\ 4 & -1\end{pmatrix}.$$

(a) Find all the eigenvalues of A.

Solution: The eigenvalues are the roots of the equation

$$\begin{aligned}0 = \det(A-zI) &= \det\begin{pmatrix}3-z & 3\\ 4 & -1-z\end{pmatrix}\\ &= z^2-2z-15 = (z+3)(z-5)\,.\end{aligned}$$

The eigenvalues are therefore -3 and 5.

(b) For each eigenvalue of A find an eigenvector.

Solution: A vector v is an eigenvector of A corresponding to an eigenvalue z provided it is a nonzero solution of $(A-zI)v = 0$. The approach taken in the book is to directly solve these equations.

An eigenvector of A corresponding to $z = -3$ satisfies

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3+3 & 3 \\ 4 & -1+3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} .$$

This leads to the equation $v_2 = -2v_1$. An eigenvector of A corresponding to the eigenvalue -3 is thereby

$$v = \begin{pmatrix} 1 \\ -2 \end{pmatrix} ,$$

or any nonzero multiple of it.

An eigenvector of A corresponding to $z = 5$ satisfies

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3-5 & 3 \\ 4 & -1-5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 4 & -6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} .$$

This leads to the equation $2v_1 = 3v_2$. An eigenvector of A corresponding to the eigenvalue 8 is thereby

$$v = \begin{pmatrix} 3 \\ 2 \end{pmatrix} ,$$

or any nonzero multiple of it.

The approach taken in class was based on the Cayley-Hamilton theorem, which states that

$$0 = p(A) = (A + 3I)(A - 5I) = (A - 5I)(A + 3I) .$$

Hence, any nonzero column of $(A - 5I)$ is an eigenvector of $z = -3$ while any nonzero column of $(A + 3I)$ is an eigenvector of $z = 5$. Because

$$A - 5I = \begin{pmatrix} -2 & 3 \\ 4 & -6 \end{pmatrix} , \qquad A + 3I = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix} ,$$

you can read off that an eigenvector of A corresponding to the eigenvalue -3 is

$$v = \begin{pmatrix} 1 \\ -2 \end{pmatrix} ,$$

or any nonzero multiple of it, while an eigenvector of A corresponding to the eigenvalue 8 is

$$v = \begin{pmatrix} 3 \\ 2 \end{pmatrix} ,$$

or any nonzero multiple of it.

(3) (10 points) Consider the linear algebraic system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 2 , \\ 2x_1 + x_2 + x_3 &= 1 , \\ x_1 - x_2 + 2x_3 &= -1 . \end{aligned}$$

Either find its general solution or else show that it has no solution.

Solution: First, a remark. By any method you can compute

$$\det \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} = 0\,.$$

This means that either there is no solution or there are many solutions. Whether there is a solution or not depends on the right-hand side of the system.

In fact, the given system has many solutions. There are many ways to see this. One of the simplest is to notice that the numbers on its right-hand side are exactly the coefficients of the x_2 variable that appear on its left-hand side. The system therefore clearly has the solution $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, and hence clearly has many solutions.

One can then find a general solution by noticing that the sum of the second and third columns of the coefficient matrix equals the first column. This means that

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 0\,.$$

It is clear that the general solution of the associated homogeneous system is given by scalar multiples of this vector. A general solution of the given system is therefore seen to be

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -s \\ 1+s \\ s \end{pmatrix} \,,$$

where s is any real number.

Alternatively, you can try to solve the given system by applying elementary row operations to the augmented matrix. This gives

$$\begin{aligned} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 2 & -1 \end{array}\right) &\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -3 & 3 & -3 \\ 0 & -3 & 3 & -3 \end{array}\right) \\ &\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \\ &\sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \,. \end{aligned}$$

A general solution can therefore be read off as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -s \\ 1+s \\ s \end{pmatrix} \,,$$

where s is any real number.

(4) (20 points) Solve each of the following initial-value problems.

(a) $\dfrac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

Solution: First, you must compute the characteristic polynomial $p(z)$ of the coefficient matrix

$$A = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}.$$

Because A is 2×2 this can either be done as

$$\begin{aligned} p(z) &= z^2 - \mathrm{tr}(A)\, z + \det(A) \\ &= z^2 - (2-1)z + (-2-10) = z^2 - z - 12\,, \end{aligned}$$

or as

$$\begin{aligned} p(z) &= \det(A - zI) \\ &= \det\begin{pmatrix} 2-z & 2 \\ 5 & -1-z \end{pmatrix} = (2-z)(-1-z) - 10 \\ &= z^2 - (2-1)z - 2 - 10 = z^2 - z - 12\,. \end{aligned}$$

By either method, after factoring you obtain

$$p(z) = (z+3)(z-4)\,,$$

whereby the eigenvalues of A are -3 and 4.

Now there are several approaches you can take. The approach used in class goes as follows. First, compute the exponential matrix $\exp(At)$. Because A has two simple real roots one has that

$$\begin{aligned} \exp(At) &= \frac{1}{4-(-3)}\left[(4I - A)\, e^{-3t} + (A + 3I)e^{4t}\right] \\ &= \frac{1}{7}\left[\begin{pmatrix} 2 & -2 \\ -5 & 5 \end{pmatrix} e^{-3t} + \begin{pmatrix} 5 & 2 \\ 5 & 2 \end{pmatrix} e^{4t}\right]. \end{aligned}$$

The solution of the initial-value problem is then given by

$$\begin{aligned} x(t) &= \exp(At)\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \frac{1}{7}\left[\begin{pmatrix} 2 & -2 \\ -5 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 5 & 2 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}\right] \\ &= \frac{1}{7}\left[\begin{pmatrix} 4 \\ -10 \end{pmatrix} e^{-3t} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} e^{4t}\right]. \end{aligned}$$

The approach used in the book goes as follows. First compute eigenvectors associated with the eigenvalues -3 and 4 respectively:

$$\begin{pmatrix} 2 \\ -5 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

A general solution is therefore found to be

$$x(t) = c_1\begin{pmatrix} 2 \\ -5 \end{pmatrix} e^{-3t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}.$$

The initial condition then leads to the equations

$$2c_1 + c_2 = 1\,, \qquad -5c_1 + c_2 = -1\,.$$

These are then solved to find $c_1 = 2/7$ and $c_2 = 3/7$. Hence,

$$x(t) = \frac{2}{7}\begin{pmatrix} 2 \\ -5 \end{pmatrix} e^{-3t} + \frac{3}{7}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}\,.$$

Notice that the details of finding the eigenvectors and of solving for c_1 and c_2 are omitted above.

(b) $\dfrac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\,, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\,.$

Solution: First, you must compute the characteristic polynomial $p(z)$ of the coefficient matrix

$$A = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}\,.$$

Because A is $2{\times}2$ this can either be done as

$$\begin{aligned} p(z) &= z^2 - \mathrm{tr}(A)\,z + \det(A) \\ &= z^2 - (1+1)z + (1+4) = z^2 - 2z + 1 + 4 \\ &= (z+1)^2 + 4\,, \end{aligned}$$

or as

$$\begin{aligned} p(z) &= \det(A - zI) \\ &= \det\begin{pmatrix} 1-z & 1 \\ -4 & 1-z \end{pmatrix} = (z-1)^2 + 4\,. \end{aligned}$$

By either method, the eigenvalues of A are seen to be $1+i2$ and $1-i2$.

Now there are several approaches you can take. The approach used in class goes as follows. First, compute the exponential matrix $\exp(At)$. Because A has a conjugate pair of simple complex roots one has that

$$\begin{aligned} \exp(At) &= I\, e^t \cos(2t) + (A - I)\, e^t \frac{\sin(2t)}{2} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^t \cos(2t) + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} e^t \sin(2t)\,. \end{aligned}$$

The solution of the initial-value problem is then given by

$$\begin{aligned} x(t) &= \exp(At)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t \cos(2t) + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t \sin(2t) \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t \cos(2t) + \frac{1}{2}\begin{pmatrix} 1 \\ -4 \end{pmatrix} e^t \sin(2t)\,. \end{aligned}$$

The approach used in the book goes as follows. First compute eigenvectors associated with the eigenvalues $1 + i2$ and $1 - i2$ respectively:

$$\begin{pmatrix} 1 \\ i2 \end{pmatrix}\,, \quad \text{and} \quad \begin{pmatrix} 1 \\ -i2 \end{pmatrix}\,.$$

The details of finding these eigenvectors are omitted here. Two linearly independent real-valued solutions are then given by

$$\begin{aligned}
\mathrm{Re}\Big(\begin{pmatrix} 1 \\ i2 \end{pmatrix} e^{(1+i2)t} \Big) &= \mathrm{Re}\Big(\begin{pmatrix} 1 \\ i2 \end{pmatrix} e^{t} \big(\cos(2t) + i \sin(2t) \big) \Big) \\
&= e^{t} \begin{pmatrix} \cos(2t) \\ -2\sin(2t) \end{pmatrix} , \\
\mathrm{Im}\Big(\begin{pmatrix} 1 \\ i2 \end{pmatrix} e^{(1+i2)t} \Big) &= \mathrm{Im}\Big(\begin{pmatrix} 1 \\ i2 \end{pmatrix} e^{t} \big(\cos(2t) + i \sin(2t) \big) \Big) \\
&= e^{t} \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix} .
\end{aligned}$$

A general solution is therefore found to be

$$x(t) = c_1 e^{t} \begin{pmatrix} \cos(2t) \\ -2\sin(2t) \end{pmatrix} + c_2 e^{t} \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix} .$$

The initial condition then leads to the equations

$$c_1 = 1 , \qquad 2c_2 = 1 ,$$

These are solved to find $c_1 = 1$ and $c_2 = -1/2$. Hence,

$$x(t) = e^{t} \begin{pmatrix} \cos(2t) + \frac{1}{2}\sin(2t) \\ \cos(2t) - 2\sin(2t) \end{pmatrix} .$$

(5) (20 points) Find a greneral solution for each of the following systems.

(a) $\dfrac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Solution: The characteristic polynomial of the coefficient matrix A is

$$p(z) = z^2 - 2z + 1 = (z-1)^2 .$$

This has the double real root $z = 1, 1$.

The approach used in class goes as follows. First, compute the exponential matrix $\exp(At)$. Because A has the simple conjugate pair of roots $2 \pm i2$ one has that

$$\begin{aligned}
\exp(At) &= I e^{t} + (A - I) t\, e^{t} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{t} + \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} t\, e^{t} .
\end{aligned}$$

A general solution is therefore

$$\begin{aligned}
x(t) &= \exp(At) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{t} + \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} t\, e^{t} \\
&= \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{t} + \begin{pmatrix} 2c_1 - 4c_2 \\ c_1 - 2c_2 \end{pmatrix} t\, e^{t} .
\end{aligned}$$

(b) $\displaystyle \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Solution: The characteristic polynomial of the coefficient matrix A is

$$p(z) = z^2 + 16\,.$$

This has the conjugate pair of simple complex roots $z = \pm i4$.

Now there are several approaches you can take. The approach used in class goes as follows. First, compute the exponential matrix $\exp(At)$. Because A has the conjugate pair of simple complex roots $\pm i4$ one has that

$$\begin{aligned} \exp(At) &= I\cos(4t) + A\frac{\sin(4t)}{4} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(4t) + \frac{1}{4}\begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}\sin(4t)\,. \end{aligned}$$

A general solution is therefore

$$\begin{aligned} x(t) &= \exp(At)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\cos(4t) + \frac{1}{4}\begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\sin(4t) \\ &= \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\cos(4t) + \frac{1}{4}\begin{pmatrix} 2c_1 - 5c_2 \\ 4c_1 - 2c_2 \end{pmatrix}\sin(4t)\,. \end{aligned}$$

The approach used in the book goes as follows. First compute eigenvectors associated with the eigenvalues $i4$ and $-i4$ respectively:

$$\begin{pmatrix} 1 + i2 \\ 2 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 - i2 \\ 2 \end{pmatrix}.$$

The details of finding these eigenvectors are omitted here. Two linearly independent real-valued solutions are then given by

$$\begin{aligned} \mathrm{Re}\left(\begin{pmatrix} 1 + i2 \\ 2 \end{pmatrix} e^{i4t}\right) &= \mathrm{Re}\left(\begin{pmatrix} 1 + i2 \\ 2 \end{pmatrix}\big(\cos(4t) + i\sin(4t)\big)\right) \\ &= \begin{pmatrix} \cos(4t) - 2\sin(4t) \\ 2\cos(4t) \end{pmatrix}, \\ \mathrm{Im}\left(\begin{pmatrix} 1 + i2 \\ 2 \end{pmatrix} e^{i4t}\right) &= \mathrm{Im}\left(\begin{pmatrix} 1 + i2 \\ 2 \end{pmatrix}\big(\cos(4t) + i\sin(4t)\big)\right) \\ &= \begin{pmatrix} 2\cos(4t) + \sin(4t) \\ 2\sin(4t) \end{pmatrix}. \end{aligned}$$

A general solution is therefore

$$x(t) = c_1\begin{pmatrix} \cos(4t) - 2\sin(4t) \\ 2\cos(4t) \end{pmatrix} + c_2\begin{pmatrix} 2\cos(4t) + \sin(4t) \\ 2\sin(4t) \end{pmatrix}.$$

(6) (12 points) Sketch phase-plane portraits for each of the following systems. State the type and stability of the origin. (Notice that these systems also appear in Problems 4 and 5.)

(a) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Solution: From Problem (4a) you know that the coefficient matrix has eigenvalues -3 and 4. The origin is therefore a *saddle point*, and hence is *unstable*.
The eigenvectors corresponding to the eigenvalues -3 and 4 are respectively

$$\begin{pmatrix} 2 \\ -5 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The phase portrait is attracting along the line $y = -\frac{5}{2}x$ and repelling along the line $y = x$.

(b) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Solution: From Problem (4b) you know that the coefficient matrix has eigenvalues $1+i2$ and $1-i2$. The origin is therefore a *spiral source*, and hence is *unstable*. Because the coefficient matrix entry $a_{12} = 1$ is positive, the orbits go clockwise around the origin.

(c) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Solution: From Problem (5b) you know that the coefficient matrix has eigenvalues $i4$ and $-i4$. The origin is therefore a *center*, and hence is *stable*. Because the coefficient matrix entry $a_{12} = -5$ is negative, the orbits go counterclockwise around the origin.

(7) (9 points) Consider the system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - xy \\ 4y - xy - 2y^2 \end{pmatrix}.$$

(a) Find all of its equilibrium points.

Solution: Equilibrium points satisfy

$$x(1-y) = 0\,, \qquad (4 - x - 2y)y = 0\,.$$

The first equation above shows that either $x = 0$ or $y = 1$.
When $x = 0$ the second equation shows that

$$(4 - 2y)y = 0\,,$$

whereby either $y = 0$ or $y = 2$. Hence, $(0,0)$ and $(0,2)$ are equilibrium points.
When $y = 1$ the second equation shows that

$$(4 - x - 2) = 0\,,$$

whereby $x = 2$. Hence, $(2,1)$ is also an equilibrium point.

The equilibrium points of the system are therefore

$$(0,0)\,, \qquad (0,2)\,, \qquad (2,1)\,.$$

(b) Compute the coefficient matrix of the linearization associated with each equilibrium point.

Solution: Because

$$\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} x - xy \\ 4y - xy - 2y^2 \end{pmatrix}\,,$$

the matrix of partial derivatives is

$$\begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} = \begin{pmatrix} 1 - y & -x \\ -y & 4 - x - 4y \end{pmatrix}\,.$$

Evaluating this matrix at each equilibrium point yields the coefficient matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \qquad \text{at } (0,0)\,,$$

$$A = \begin{pmatrix} -1 & 0 \\ -2 & -4 \end{pmatrix} \qquad \text{at } (0,2)\,,$$

$$A = \begin{pmatrix} 0 & -2 \\ -1 & -2 \end{pmatrix} \qquad \text{at } (2,1)\,.$$

This is all that is asked of you. However, if you had been asked to classify the type and stability of each critical point then you can easily see that $(0,0)$ is a nodal source (the eigenvalues are 1 and 4) and is thereby unstable, $(0,2)$ is a nodal sink (the eigenvalues are -1 and -4) and is thereby asymptotically stable, and $(2,1)$ is a saddle (the eigenvalues are $-1-\sqrt{3}$ and $-1+\sqrt{3}$) and is thereby unstable.

(8) (9 points) Suppose you know that for some nonlinear system of differential equations

- the equilibrium solutions are $(0,0)$, $(4,-2)$, and $(4,2)$;
- for $(0,0)$ the linearization has eigenvalues -2 and -1 with respective eigenvectors

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}\,;$$

- for $(4,-2)$ the linearization has eigenvalues 2 and 1 with respective eigenvectors

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,;$$

- for $(4,2)$ the linearization has eigenvalues 1 and -1 with respective eigenvectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,;$$

Sketch a plausible phase portrait for the system. Identify the type and stability of each equilibrium solution.

Solution: The type and stability of each equilibrium solution is determined as follows.

- The equilibrium solution (0,0) has two negative simple real eigenvalues. It therefore is a *nodal sink* and is thereby asymptotically stable.
- The equilibrium solution (4,-2) has two positive simple real eigenvalues. It therefore is a *nodal source* and is thereby unstable.
- The equilibrium solution (4,2) has one negative and one positive simple real eigenvalue. It therefore is a *saddle* and is thereby unstable.

There are many plausible phase portraits that one might draw. Some will be sketched during class. They all share the following features.

- Near the *nodal sink* $(0,0)$ there is one orbit that approaches $(0,0)$ tangent to each side of the line $y = -x$. Every other orbit approaches $(0,0)$ tangent to the line $y = x$.
- Near the *nodal source* $(4,-2)$ there is one orbit that emerges from $(4,-2)$ tangent to each side of the line $y = -x + 2$. Every other orbit emerges from $(4,-2)$ tangent to the line $x = 4$.
- Near the *saddle* $(4,2)$ there is one orbit that emerges from $(4,2)$ tangent to each side of the line $y = x - 2$. There is also one orbit that approaches $(4,2)$ tangent to each side of the line $x = 4$.