Math 246, Sample Problems for First In-Class Exam

The exam is closed book, closed notes, and no calculators are allowed. Work at most one problem on each page of your exam booklet, marking the problem number at the top of the page. Your answer(s) to each part should be circled, and you must show your work. Brief descriptions of your reasoning are helpful, especially for earning partial credit. Any work that you do not want to be considered should be crossed out.

1. (a) Write a MATLAB command that evaluates the definite integral

$$\int_0^\infty \frac{r}{1+r^4}\, dr \,.$$

   (b) Sketch the graph that you expect would be produced by the following MATLAB commands.

   [X, Y] = meshgrid(–5:0.2:5,–5:0.2:5)
   contour(X, Y, X.^2 + Y.^2, [25, 25])
   axis square

2. Give an explicit solution to each of the following initial value problems.

   (a) $\dfrac{dz}{dt} = \dfrac{\cos(t) - z}{1+t}\,, \quad z(0) = 2\,.$

   (b) $\dfrac{du}{dz} = e^u + 1\,, \quad u(0) = 0\,.$

3. Consider the differential equation

$$\frac{dy}{dt} = y^3 - 4y\,.$$

   (a) Find all of the equilibrium (stationary) solutions and classify each as stable or unstable.

   (b) Draw a graph of y versus t showing the direction field and several solution curves, including all of the equilibrium solutions and solutions above and below each equilibrium value.

   (c) If $y(0) = 1$, what is the limiting value of y as $t \to \infty$?

4. Suppose you are using the fourth-order Runge-Kutta method to numerically solve an initial-value problem over the interval [0,10]. By what factor would you expect the error to decrease when you increase the number of steps taken from 100 to 300?

5. A tank initially contains 100 liters of pure water. Beginning at $t = 0$ brine (salt water) with a salt concentration of 2 grams per liter (g/l) flows enters the tank at a constant rate of 3 liters per minute (l/min) and the well-stirred mixture leaves the tank at the same rate. Let $S(t)$ denote the mass of salt in the tank at time $t \geq 0$.
   (a) Is $S(t)$ an increasing or decreasing function of t? (Give your reasoning.)
   (b) What is the behavior of $S(t)$ as $t \to \infty$? (Give your reasoning.)
   (c) Derive a formula for $S(t)$.

6. Give an implicit general solution to each of the following differential equations.
   (a) $\left(\dfrac{y}{x} + 3x\right)dx + \left(\ln(x) - y\right)dy = 0\,.$
   (b) $(x^2 + y^3 + 2x)dx + 3y^2dy = 0\,.$

7. A 2 kilogram (kg) mass initially at rest is dropped in a medium that offers a resistance of $v^2/40$ newtons (= kg m/sec^2) where v is the downward velocity of the mass in meters per second. The gravitational acceleration is 9.8 m/sec^2.
   (a) What is the terminal velocity of the mass?
   (b) Write down an initial value problem that governs v as a function of time. (You do not have to solve it!)

8. Consider the following MATLAB function M-file.

```
function [t,y] = solveit(ti, yi, tf, n)

h = (tf – ti)/n;
t = zeros(n + 1, 1);
y = zeros(n + 1, 1);
t(1) = ti;
y(1) = yi;
for i = 1:n
z = t(i)^2 + y(i)^2;
t(i + 1) = t(i) + h;
y(i + 1) = y(i) + (h/2)*(z + t(i + 1)^2 + (y(i) + h*z)^2);
end
```

   (a) What is the initial-value problem being solved numerically?
   (b) What is the numerical method being used to solve it?
   (c) What are the output values of t(2) and y(2) that you would expect for input values of ti = 1, yi = 1, tf = 5, n = 20 for this method?