Math 246, Sample Problems for Second In-Class Exam

Exam ends at 5 minutes of the hour, no later. Exam is closed book and closed notes. No calculators or other electronic devices are allowed. Work at most one problem on each page of your examination booklet, marking the problem number at the top of the page. Your answer(s) to each part should be circled, and you must show your work. Brief descriptions of your reasoning are helpful, especially for earning partial credit. Any work that you do not want to be considered should be crossed out.

(1) Let L be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (shown with multiplicities) are $-2+3i$, $-2-3i$, $7i$, $7i$, $-7i$, $-7i$, 5, 5, -3, 0, 0, 0.
  (a) What is the order of L?
  (b) Give a general real solution of the homogeneous equation $Ly = 0$?

(2) Solve each of the following initial-value problems.
  (a) $y'' + 4y' + 4y = 0\,, \quad y(0) = 1\,, \quad y'(0) = 0\,.$
  (b) $y'' + y = 4e^t\,, \quad y(0) = 0\,, \quad y'(0) = 0\,.$

(3) Find a general solution for each of the following equations.
  (a) $y'' + 4y' + 5y = 3\cos(2t)\,.$
  (b) $y'' - y = e^t\,.$

(4) Given that x and x^2 are linearly independent solutions of the homogeneous equation

$$x^2y'' - 2xy' + 2y = 0\,, \qquad x > 0\,,$$

find a general solution of the equation

$$x^2y'' - 2xy' + 2y = xe^x\,, \qquad x > 0\,.$$

You may express the solution in terms of definite integrals.

(5) What answer will be produced by the following MATLAB commands?

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>> ode1 = 'D2y + 2*Dy + 5*y = 16*exp(t)';
>> dsolve(ode1, 't')
ans =
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(6) The vertical displacement of a mass on a spring is given by

$$z(t) = \sqrt{3}\cos(2t) + \sin(2t)\,.$$

Express this in the form $z(t) = A\cos(\omega t - \delta)$, identifying the amplitude and phase of the oscillation.

(7) When a mass of 4 grams is hung vertically from a spring, at rest it stretches the spring 9.8 cm. (Gravitational acceleration is $g = 980$ cm/sec^2.) At $t = 0$ the mass is displaced 3 cm above its equilibrium position and released with no initial velocity. It moves in a medium that imparts a drag force of 2 dynes (1 dyne = 1 gram cm/sec^2) when the speed of the mass is 4 cm/sec. There are no other forces. (As usual, assume the spring force is proportional to displacement and the drag force is proportional to velocity.)

- (a) Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve the initial-value problem, just write it down!)
- (b) What is the natural frequency of the spring?
- (c) Is the spring over damped, critically damped, or under damped? Why?

(8) Compute the Laplace transform of $f(t) = te^{3t}$ from its definition.

(9) Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y'' + 4y' + 13y = f(t)\,, \qquad y(0) = 4\,, \quad y'(0) = 1\,,$$

where

$$f(t) = \begin{cases} \cos(t) & \text{for } 0 \le t < 2\pi\,, \\ 0 & \text{for } t \ge 2\pi\,. \end{cases}$$

You may refer to the table on the last page. (DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$.)

(10) Find the inverse Laplace tramsform of the following functions:

- (a) $F(s) = \dfrac{2}{(s+5)^3}$,
- (b) $F(s) = \dfrac{3s}{s^2 - s - 6}$,
- (c) $F(s) = \dfrac{(s-2)e^{-3s}}{s^2 - 4s + 5}$.

You may refer to the table below.

A Short Table of Laplace Transforms

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad \text{for } s > 0\,.$$

$$\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2 + b^2} \qquad \text{for } s > 0\,.$$

$$\mathcal{L}\{\sin(bt)\} = \frac{b}{s^2 + b^2} \qquad \text{for } s > 0\,.$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \qquad \text{where } F(s) = \mathcal{L}\{f(t)\}\,.$$

$$\mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs}F(s) \qquad \text{where } F(s) = \mathcal{L}\{f(t)\}$$

and u is the step function.