

Numerical Analysis I: AMSC/CMSC 666
Homework 1, due Friday, 18 February 2005

- (1) Recall Rolle's Theorem states if $g \in C([x_L, x_R]) \cap C^1((x_L, x_R))$ has two zeros counting multiplicity in $[x_L, x_R]$ then g' has at least one zero in (x_L, x_R) . Prove the following corollary of Rolle's Theorem.

If $g \in C^{k-1}([x_L, x_R])$ has k zeros counting multiplicity in $[x_L, x_R]$ then g' has at least $k - 1$ zeros counting multiplicity in $[x_L, x_R]$.

(Recall that x_o is a zero of g with multiplicity m provided $g^{(l)}(x_o) = 0$ for every $l = 0, 1, \dots, m - 1$.)

- (2) Let $\{x_1, x_2, \dots, x_m\}$ be an ordered set of m distinct points in \mathbb{R} and $\{n_1, n_2, \dots, n_m\}$ be an associated set of m positive integers. Set

$$\begin{aligned}\Pi(x) &= \prod_{i=1}^m (x - x_i)^{n_i}, \\ \Pi_k(x) &= \prod_{\substack{i=1 \\ i \neq k}}^m (x - x_i)^{n_i} = \frac{\Pi(x)}{(x - x_k)^{n_k}}, \\ n &= \sum_{i=1}^m n_i.\end{aligned}$$

Suppose that $f \in C^n([x_L, x_R])$ and the polynomial $p[f]$ is defined by

$$p[f](x) = \sum_{k=1}^m \Pi_k(x) T_{x_k}^{n_k-1} \left[\frac{f}{\Pi_k} \right] (x).$$

Here $T_{x_k}^{n_k-1}[g]$ denotes the Taylor approximation of g of order $n_k - 1$ centered at x_k , which is formally given by

$$\begin{aligned}T_{x_k}^{n_k-1}[g](x) &= g(x_k) + (x - x_k)g'(x_k) + \dots \\ &\quad + \frac{1}{(n_k - 1)!} (x - x_k)^{n_k-1} g^{(n_k-1)}(x_k).\end{aligned}$$

Show that $p[f]$ is a polynomial of degree at most $n - 1$ that satisfies the n generalized Hermite interpolation conditions

$$p^{(l)}(x_k) = f^{(l)}(x_k) \quad \text{for } k = 1, \dots, m \text{ and } 0 \leq l < n_k.$$

- (3) Let $\{x_1, x_2, \dots, x_m\}$ be an ordered set of m distinct points in $[x_L, x_R]$. Let $f \in C^{2m}([x_L, x_R])$ and set

$$p(x) = P[f; x_1, x_1, x_2, x_2, \dots, x_m, x_m](x),$$

where each x_k appears in the argument list exactly twice.

- (a) Give a uniform bound on $p(x) - f(x)$ over $[x_L, x_R]$ in the form

$$\|p - f\|_\infty \leq M_m \|f^{(2m)}\|_\infty.$$

- (b) For what choice of $\{x_1, x_2, \dots, x_m\}$ will M_m be minimum? What is the associated minimum value of M_m ?

- (4) Let $f(x) = \sin(x)$. Give the cubic Chebyshev interpolation of f over $[-\pi, \pi]$. Give a uniform bound on the error of this approximation over $[-\pi, \pi]$.

- (5) Define the inner product $(\cdot | \cdot)$ on $C([-1, 1])$ by

$$(f | g) = \frac{1}{\pi} \int_{-1}^1 \frac{f(x)g(x)}{\sqrt{1-x^2}} dx.$$

Show that the Chebyshev polynomials T_n are mutually orthogonal with respect to this inner product.

Project. Consider the function $f(x) = (e^x - 1)/(e^x + 1)$ over $[-5, 5]$. Plot a graph that compares the following interpolations of f over $[-5, 5]$:

- the interpolation $P[f; -5, -3, -1, 1, 3, 5](x)$,
- the interpolation $P[f; -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5](x)$,
- the interpolation $P[f; -5, -5, -3, -3, -1, -1, 1, 1, 3, 3, 5, 5](x)$,
- the Chebyshev interpolation of degrees 5 over $[-5, 5]$,
- the Chebyshev interpolation of degrees 10 over $[-5, 5]$,

Explain the differences you see. Give the coefficients of the Newton form of each interpolating polynomial. You might find it helpful to program the divided differences algorithm (Stoer and Bulirsch section 2.1.3) to generate the Newton form of the polynomials asked for above.

Plot a graph that compares the errors of the above interpolations over $[-5, 5]$. How do the errors you see compare with the uniform error estimate given in class. Explain the differences you see.