

Numerical Analysis I: AMSC/CMSC 666
Homework 2, due Friday, 4 March 2005

- (1) Let X be a real inner product space with inner product $(\cdot | \cdot)$. Let $\{v_n\}_{n=1}^\infty \subset X$ be linearly independent. Let $\gamma_{mn} = (v_m | v_n)$ be the entries of the associated Gram matrix. Set $\delta_0 = 1$ and

$$\delta_n = \det \begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nn} \end{pmatrix}.$$

for every $n = 1, 2, \dots$.

- (a) Show that every δ_n is positive.
 (b) Consider $\{u_n\}_{n=1}^\infty \subset X$ where u_n is defined by

$$u_n = \frac{1}{\sqrt{\delta_{n-1}\delta_n}} \det \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1(n-1)} & v_1 \\ \vdots & \ddots & \vdots & \vdots \\ \gamma_{(n-1)1} & \cdots & \gamma_{(n-1)(n-1)} & v_{n-1} \\ \gamma_{n1} & \cdots & \gamma_{n(n-1)} & v_n \end{pmatrix}.$$

Show that the set $\{u_n\}_{n=1}^\infty$ is orthonormal and that

$$\text{span}\{u_1, \dots, u_n\} = \text{span}\{v_1, \dots, v_n\}.$$

- (c) Explain why the above recipe is not better than the Gram-Schmidt algorithm for numerically generating the orthonormal set $\{u_n\}_{n=1}^\infty$.

- (2) Let $\{p_n(x)\}_{n=0}^\infty$ be the sequence of monic polynomials orthogonal over $(0, \infty)$ with respect to the real inner product

$$(f | g) = \int_0^\infty f(x) g(x) x e^{-x} dx,$$

where p_n has degree n . Find p_n for $n = 0, 1, 2, 3, 4$.

- (3) Let $f \in C^m(\mathbb{S}^1; \mathbb{C})$. Let

$$S^n f(x) = \sum_{k=-n}^n \hat{f}_k e^{ikx},$$

where

$$\hat{f}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x) dx.$$

Show there exists an $M < \infty$ such that

$$\|S^n f - f\|_\infty \leq \frac{M}{n^{m-1}} \quad \text{for every } n \in \mathbb{N}.$$

Here $\| \cdot \|_\infty$ is the uniform norm over \mathbb{S}^1 .

- (4) Partition the interval $[x_L, x_R]$ into n subintervals as

$$x_L = x_0 < x_1 < x_2 < \cdots < x_{n-2} < x_{n-1} < x_n = x_R.$$

Prove that the linear spline minimizes the integral

$$\int_{x_L}^{x_R} |Y'(x)|^2 dx,$$

subject to the constraints $Y(x_i) = y_i$ for $i = 0, \dots, n$ over the class of functions Y that are continuous over $[x_L, x_R]$ and are smooth over the subinterval (x_{i-1}, x_i) for every $i = 1, \dots, n$.

- (5) Partition the interval $[x_L, x_R]$ into n subintervals as

$$x_L = x_0 < x_1 < x_2 < \cdots < x_{n-2} < x_{n-1} < x_n = x_R.$$

Find the quadratic spline Y that interpolates the data

$$\frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} Y(x) dx = y_i \quad \text{for } i = 1, \dots, n,$$

and that satisfies $Y'(x_L) = \dot{y}_L$ and $Y'(x_R) = \dot{y}_R$.

Project. Consider the 2π -periodic function

$$f(x) = \text{sign}(\cos(x)).$$

Give an analytic expression for \hat{f}_k . Plot a graph over $(-\pi, \pi)$ that compares f , $S^4 f$, $S^8 f$, and $S^{16} f$. Explain what you see in neighborhoods $x = \pm \frac{\pi}{2}$, which is where f has jump discontinuities.