Numerical Analysis I: AMSC/CMSC 666
Homework 3, due Friday, 18 March 2005

(1) Consider numerical approximation of the integral

$$I(f) = \int_{x_L}^{x_R} f(x)\, \mathrm{d}x \, .$$

Let $Q_n(f)$ denote quadrature by the midpoint rule with n uniform subintervals. The error can be expressed formally as

$$E_n(f) = Q_n(f) - I(f) = \int_{x_L}^{x_R} K(x) f''(x)\, \mathrm{d}x \, ,$$

where K is the Peano kernel.

(a) Derive an explicit expression for the Peano kernel and show that it is nonpositive.

(b) Derive a sharp bound of the form

$$\big|E_n(f)\big| \leq C_n \big\|f''\big\|_\infty \, ,$$

that holds for every $f \in C^2([x_L, x_R])$. A bound is "sharp" whenever the inequality is an equality for some nonzero f.

(c) Derive a sharp bound of the form

$$\big|E_n(f)\big| \leq M_n \int_{x_L}^{x_R} |f''(x)|\, \mathrm{d}x \, ,$$

that holds for every $f \in C^1([x_L, x_R])$ such that f'' is bounded and is continuous at all but a finite number of points.

(2) Let $Q_\Delta(f)$ denote quadrature over the interval $[x_L, x_R]$ by the midpoint rule with uniform subintervals of length Δ. Show that as Δ becomes smaller $Q_\Delta(f)$ has the asymptotic behavior

$$Q_\Delta(f) = I(f) - C\Delta^2\big(f'(x_R) - f'(x_L)\big) + O\big(\Delta^4\big) \, ,$$

for some positive constant C.

(3) Let $Q_\Delta(f)$ denote quadrature over an interval by the trapezoidal rule with uniform subintervals of length Δ. Use the Euler-Maclaurin formula to extrapolate $Q_\Delta(f)$, $Q_{2\Delta}(f)$, $Q_{3\Delta}(f)$, and $Q_{6\Delta}(f)$ to obtain an eighth order accurate quadrature.

(4) Find $\{p_0, p_1, p_2\}$ such that p_i is a monic polynomial of degree i and this set is orthongonal over $(0, \infty)$ with respect to the weight $w(x) = e^{-x}$. Find the nodes and weights of the two-point Gaussian quadrature formula

$$\int_0^\infty f(x)\, e^{-x} \mathrm{d}x \approx w_1 f(x_1) + w_2 f(x_2)\,.$$

Give a bound on the error of this formula.

(5) Derive the one-, two-, three-, and four-point Gaussian quadrature formulas such that

$$\int_{-1}^1 f(x) x^4 \,\mathrm{d}x \approx \sum_{j=1}^n f(x_j)\, w_j\,.$$

Give bounds on the error of these formulas.

Project. Consider the definite integrals

$$\int_0^1 e^{x^2}\,\mathrm{d}x\,, \qquad \int_0^1 |2x - 1|\,\mathrm{d}x\,.$$

Approximate these numerically by Romberg quadrature with 8 uniform subintervals. Compare the Romberg results with those obtained by trapezoidal rule, Simpson's rule, and Milne's rule with 8 uniform subintervals. Explain what you see.